The Central Limit Theorem

{The purpose of this initial exercise is to provide ample opportunity to familiarise yourself with the concept of the sampling distribution of the sample means and to verify the fact that $\mathrm{E}\left[\overline{X}\right] = \mu$ and $\mathrm{Var}\left[\overline{X}\right] = \frac{\sigma^2}{n}$ for such distributions.}

1. For each of the following distributions,

   a) find the mean μ and variance σ^2,

   b) by considering all possible samples of size 2, verify that $\mathrm{E}\left[\overline{X}\right] = \mu$ and $\mathrm{Var}\left[\overline{X}\right] = \frac{\sigma^2}{n}$.

   i)

x	0	1	2
$\mathrm{P}(X = x)$	0·6	0·3	0·1

   ii)

x	0	1	2	3
$\mathrm{P}(X = x)$	0·2	0·3	0·3	0·2

2. Find the mean μ and variance σ^2 of the random variable X shown in the table.

x	1	4	7
$\mathrm{P}(X = x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

   By considering the 27 possible samples of size 3 taken from this distribution, verify that $\mathrm{E}\left[\overline{X}\right] = \mu$ and $\mathrm{Var}\left[\overline{X}\right] = \frac{\sigma^2}{n}$.

1. Marks in a mathematics examination are normally distributed with mean 50 and standard deviation 10.

   What is the probability that:

   a) the mean mark of a group of 5 students will be above 60,

   b) the mean mark of a group of 20 students will be between 44 and 48 ?

2. The weights of a large group of animals have mean 8·2kg and variance 4·84kg^2. What is the probability that a random selection of 80 from the group will have mean weight between 8·3 and 8·4kg ?

   State any assumptions that you make.

3. The weekly wages paid to the weekly–paid staff of a manufacturing company are approximately normally distributed with a mean of £33·40 and a standard deviation of £3·00.

   a) Calculate the probability that the mean weekly wage of a random sample of 9 of the weekly–paid staff is within £2 of the population mean of £33·40.

   b) How large a random sample should be taken if the sample mean is to have probability 0·9 of being within £1 of the population mean ?

4. A random sample of size 100 is taken from Bin(20, 0·6). Find the probability that

   a) $\overline{X}$ is greater than 12·4 b) $\overline{X}$ is less than 12·2,

   where $\overline{X}$ denotes the sample mean.

5. Random samples of size 30 are taken from Poi(4·5). Find the probability that a randomly chosen sample has a sample mean in excess of 5.

6. A random samples of size 30 is taken from Poi(4). Find:
   a) P($\overline{X} < 4·5$) b) P($\overline{X} > 3·8$) c) P($3·8 < \overline{X} < 4·5$).

7. If a large number of samples, of size n, are taken from Poi(4·6) and approximately 2·5% of the sample means are less than 4·005, estimate the value of n.

   {Hint: Let $X \sim$ Poi(4·6), then $\overline{X}$ approx. $\sim$ N(4·6, $\frac{4·6}{n}$). P($\overline{X} < 4·005$) = 0·025, ……standardise etc.**}**

8. If a large number of samples, of size n, are taken from Poi(2·9) and approximately 1% of the sample means are greater than 3·41, estimate n.

9. If a large number of samples of size n are taken from Bin(20, 0·3) and approximately 90% of the sample means are less than 2·7, estimate the value of n.

10. The standard deviation of the masses of articles in a large population is 4·55kg. If random samples of size 100 are drawn from the population, find the probability that a sample mean will differ from the true population mean by less than 0·8kg.

11. The weight of luggage that aircraft passengers take with them is distributed with mean 20kg and standard deviation 5kg. A certain type of aircraft carries 100 passengers.

    What is the probability that the total weight of the passengers' luggage exceeds 2150kg ?

    {HINT:- turn the problem into one involving a sample mean and then apply the central limit theorem.**}**

Answers.

1. a) 0.0127 b) 0.1819.
2. Assume that the sampling distribution of the sampling means is approximately normal.
   0.1339.
3. a) 0.9544 b) 25.
4. a) 0.034 b) 0.8194.
5. 0.0983.
6. a) 0.9145 b) 0.7081 c) 0.6226.
7. 50.
8. 60.
9. 20.
10. 0.9212.
11. 0.00135.