OCR:– APPLICATIONS OF INTEGRATION

1) Show that the length of the arc of the curve $y = \cosh x$ between $x = 0$ and $x = \ln 3$ is $\frac{5}{3}$ units.

2) A curve (circle!) has equation given by $x^2 + y^2 = 1$. Use implicit differentiation to show that

$$1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{1}{1 - x^2}.$$

Hence show that the arc of the curve between $x = 0$ and $x = 1$ has length $\dfrac{\pi}{2}$ units.

3) Starting from the definition of $\cosh x$ in terms of exponentials, show that $\cosh 2x = 2\cosh^2 x - 1$.

Show that the length L of the arc of the curve $y^2 = 4x$, joining the points (0, 0) and (1, 2) is given by

$$\int_0^1 \sqrt{\frac{1 + x}{x}}\,\mathrm{d}x\,.$$

By using the substitution $x = \sinh^2 u$, or otherwise, show that $L = \sinh^{-1}(1) + \sqrt{2}$.

4) A curve C has parametric equations $x = t - \tanh t$, $y = \operatorname{sech} t$.

i) Show that $\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \tanh^2 t$

ii) Show that the length of the arc of the curve C between the points given by $t = 0$ and $t = \ln 2$ is $\ln\left(\frac{5}{4}\right)$ units.

5) A curve C has parametric equations $x = 4t^2$; $y = t^4 - 4\ln t$.

i) Show that $\left(\dfrac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \dfrac{16}{t^2}(t^4 + 1)^2$.

ii) Show that the length of the arc of the curve C between the points given by $t = 1$ and $t = 2$ is $15 + 4(\ln 2)$ units.

6) In polar co-ordinates the equation of a circle is given by $r = a$ where $a > 0$.
Use integration to show that the circumference of the circle has length $2\pi a$.

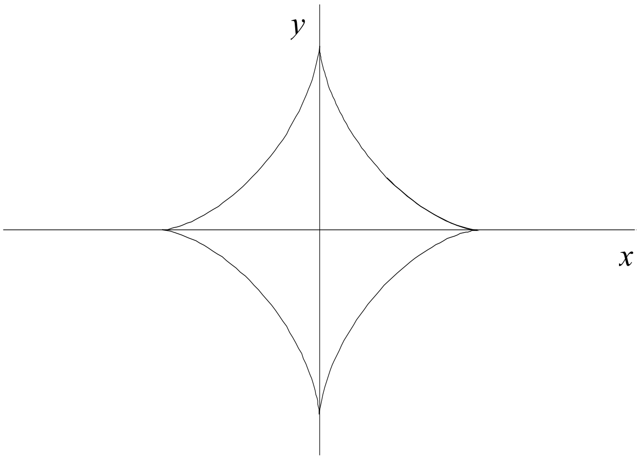
7) A curve has polar equation $r = \sin^3\left(\frac{1}{3}\theta\right)$.

i) Show that $r^2 + \left(\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \sin^4\left(\frac{1}{3}\theta\right)$

ii) Show that the arc length of the curve for $0 \le \theta \le 3\pi$, is $\frac{3}{2}\pi$.

8) Find the length of the circumference of the cardioid $r = a(1 - \cos\theta)$.

9) The curve shown in the diagram has parametric equations $x = a\cos^3 t$, $y = a\sin^3 t$, where $a > 0$ and $0 \le t < 2\pi$.



i) Show that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{9}{4}a^2\sin^2 2t$.

ii) Show that the total length of the curve is $6a$ units.

{**Hint:** use the symmetry of the curve when choosing limits for your integration.}

10) The unit circle has equation given by $x^2 + y^2 = 1$.

Show that the area of the surface formed when the arc of the circle between $x = 0$ and $x = 1$ is rotated completely about the x–axis is 2π units^2. **{This effectively proves that the surface area of a sphere is $4\pi r^2$.}**

11) i) Show that $\frac{d}{dt}(\text{sech}\, t) = -\text{sech}\, t \tanh t$.

ii) A curve C has parametric equations $x = t - \tanh t$, $y = \text{sech}\, t$.

The arc of the curve between the points with given by t = 0 and $t = \ln 2$ is rotated about the x-axis. Given that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \tanh^2 t$, show that the area of the surface formed is $\frac{2\pi}{5}$ units^2.

12) The parametric equations of a curve are $x = a\cos^3 t$, $y = a\sin^3 t$, where a is a positive constant.

Show that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9a^2\sin^2 t\cos^2 t$.

The arc of the curve between $t = 0$ and $t = \frac{1}{2}\pi$ is rotated through 2π radians about the x–axis. Find the area of the surface of revolution so formed, giving your answer in terms of a and π.

13) A curve C has parametric equations $x = 4t^2$; $y = t^4 - 4\ln t$.

Show that $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{16}{t^2}\left(t^4 + 1\right)^2$.

Find the area of the surface generated when the arc of the curve from $t = 1$ to $t = 2$ is rotated completely about **<u>the y–axis</u>**.

14) The parametric equations of a curve are $x = a(t - \sin t)$, $y = a(1 - \cos t)$, where a is a positive constant.

Show that $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 4a^2\sin^2\left(\tfrac{1}{2}t\right)$.

The arc of the curve between $t = 0$ and $t = 2\pi$ is rotated completely about the x–axis.

Show that the area of the surface of revolution formed is

$$8\pi a^2 \int_0^{2\pi} \left[1 - \cos^2\left(\tfrac{1}{2}t\right)\right]\sin\left(\tfrac{1}{2}t\right)\mathrm{d}t$$

and hence find this area.

15) i) Show that $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\operatorname{cosec}\theta\right) = -\operatorname{cosec}\theta\cot\theta$ and $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\cot\theta\right) = -\operatorname{cosec}^2\theta$.

ii) Find the area of the surface generated by rotating the arc of the curve $r = a\operatorname{cosec}\theta$ between $\theta = \tfrac{1}{4}\pi$ and $\theta = \tfrac{1}{2}\pi$ about the initial line $\theta = 0$.

16) A circle C has radius b and centre $(a, 0)$ where $0 < b < a$. A doughnut is formed by rotating C about the y-axis. By expressing the equation of c in parametric form, find the surface area of the doughnut.

17) i) Show, using the definitions of $\sinh x$ and $\cosh x$, that $\cosh 2x = 1 + 2\sinh^2 x$ and $\sinh 2x = 2\sinh x\cosh x$.

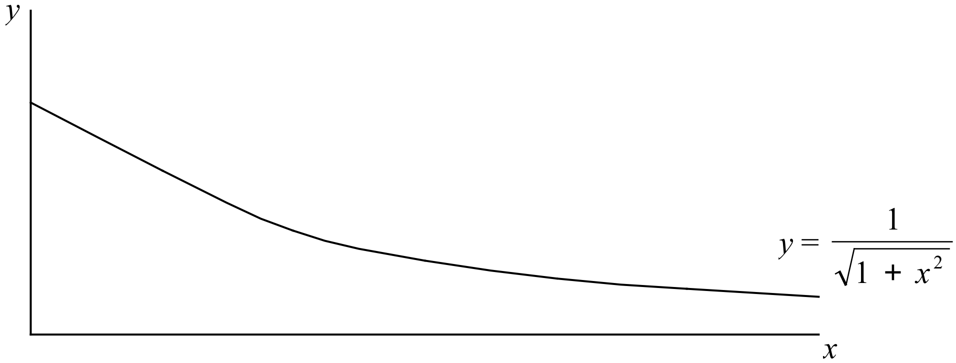
ii) By means of the substitution $x + \tfrac{1}{2} = \tfrac{1}{2}\cosh u$, or otherwise, find $\int \sqrt{x^2 + x}\,\mathrm{d}x$.

iii) The arc of the curve $y = 2\sqrt{x}$ between $x = 0$ and $x = 1$ is rotated completely about the ***y*-axis**. Find the area of the surface of revolution generated.

*18) A glass vessel has the shape of a volume of revolution obtained by rotating the region defined by $y \geq x^2$, $0 \leq y \leq \tfrac{1}{2}$, about the y-axis through two right angles, where the units are metres. The vessel contains the vapour of a chemical, the density of which at height y metres above the base of the vessel is $2\sqrt{1-2y}$ grams per metre^3. By considering the mass of vapour in a layer between heights y and $y + \delta y$ metres, show that the total mass of vapour in the vessel is $\int_0^{\frac{1}{2}} 2\pi y\sqrt{1-2y}\,dy$ grams.

Evaluate this integral, giving your answer as a rational multiple of π.

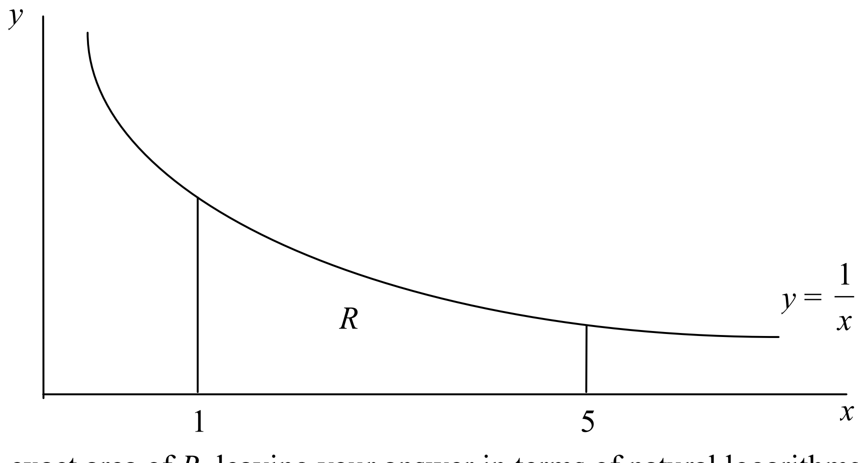
19) The diagram shows a sketch of the curve $y = \dfrac{1}{\sqrt{1+x^2}}$ for $x \geq 0$.



Copy this diagram and, by considering the areas of suitable rectangles, demonstrate that

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{10}} < \int_0^3 \frac{1}{\sqrt{1+x^2}}\,dx < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{5}}.$$

20) The diagram shows the region R bounded by the curve $y = \dfrac{1}{x}$, the x-axis and the lines $x = 1$ and $x = 5$.



Use integration to find the exact area of R, leaving your answer in terms of natural logarithms.

By considering the areas of suitable rectangles, deduce that

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < \ln 5 < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

21) Sketch the graph of $y = \dfrac{1}{x^2}$, for $x > 0$, and on your diagram draw appropriate rectangles to demonstrate the inequality

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} < \int_1^5 \frac{1}{x^2}\,dx .$$

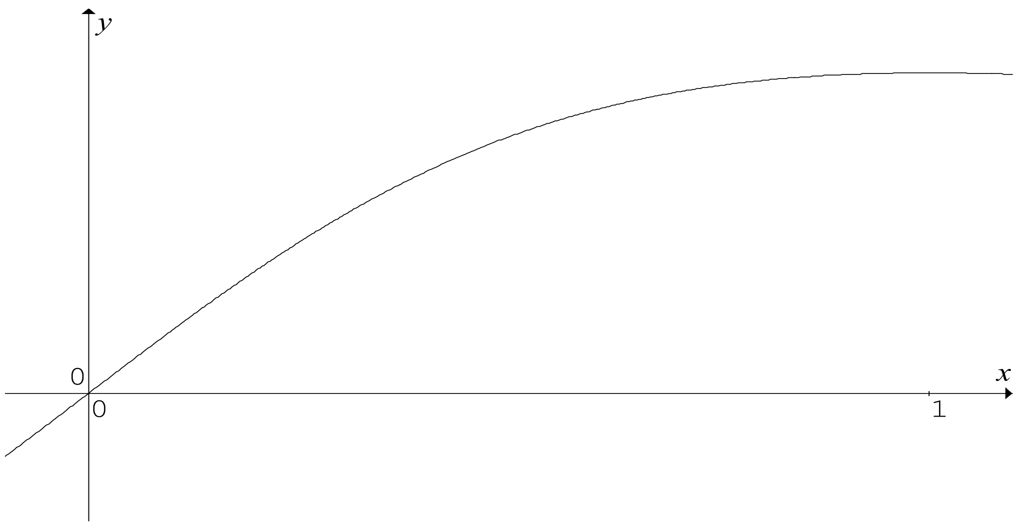
Show that $\displaystyle\sum_{r=1}^{N} \frac{1}{r^2} < 2 - \frac{1}{N}$. **{Hint:** consider $\displaystyle\int_1^N \frac{1}{x^2}\,dx$ and proceed as above.**}**

22) Rectangles of equal width h are drawn under the graph of $y = 1 + x^2$ from $x = 0$ to $x = 3$. Illustrate this with a sketch. Show that the area of the fourth rectangle is $h + 9h^3$, and find (in terms of h) the total area of the first four rectangles.

How many rectangles are there if $h = 0.1$? Show that the total area of the rectangles in this case is $3 + 0.1^3 \displaystyle\sum_{i=1}^{29} i^2$.

Find the difference between the value of this sum and the exact area under the curve from $x = 0$ to $x = 3$ found by integration.

23) The diagram shows a sketch of the graph of $y = \dfrac{x}{1 + x^2}$ for x between 0 and 1.



a) Explain in detail how $\displaystyle\sum_{r=1}^{n} \frac{r}{n^2 + r^2}$ is related to the area under the curve $y = \dfrac{x}{1 + x^2}$ between $x = 0$ and $x = 1$.

b) Evaluate the limit $L = \lim_{n\to\infty} \displaystyle\sum_{r=1}^{n} \frac{r}{n^2 + r^2}$.

c) Show that $L < \displaystyle\sum_{r=1}^{n} \frac{r}{n^2 + r^2} < L + \frac{1}{2n}$.

ANSWERS.

8) $8a$ units.

12) $\dfrac{6a^2\pi}{5}$ units^2.

13) 384π units^2.

14) $\dfrac{64\pi a^2}{3}$ units^2.

15) ii) $2\pi a^2$ units^2.

16) $4\pi^2 ab$ units^2.

17) ii) $\frac{1}{4}(2x + 1)\sqrt{x^2 + x} - \frac{1}{8}\cosh^{-1}(2x + 1) + K$, iii) $\pi\left(\dfrac{3\sqrt{2}}{2} - \dfrac{1}{4}\cosh^{-1}3\right)$ units^2.

18) $\dfrac{2}{15}\pi$.

22) $4h + 14h^3$. 30 rectangles. 0.445.

23) a) The given quantity is an approximation (over estimate) of the area under the given curve between $x = 0$ and $x = 1$.

b) $\dfrac{1}{2}\ln 2$.