

APPLICATION OF CALCULUS

{**REMEMBER**, stationary points refer to co-ordinates whilst stationary values refer to values of $y = f(x)$.}

1. Find the values of $f''(1)$ for each of the following functions.

a) $f(x) = (x - 4)^5$

b) $f(x) = \ln x$

c) $f(x) = e^{3x+1}$

d) $f(x) = \sqrt{x + 3}$.

2. Given that $y = \ln(4 + 3x)$, find $\frac{dy}{dx}$ and show that $\frac{d^2y}{dx^2} = -\frac{9}{16}$ when $x = 0$.

3. Given that $y = \ln(1 + 4x)$, show that $\frac{d^2y}{dx^2} = -\frac{16}{(1 + 4x)^2}$.

4. Find the co-ordinates of the stationary points on the following curves and investigate their nature.

a) $y = x^2 + x^3$

b) $y = x^3 - 3x + 2$

c) $y = x + \frac{1}{x}$

d) $y = x - 2\ln x$

e) $y = x^2 + \frac{54}{x}$

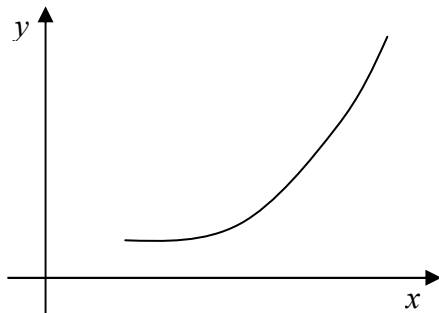
f) $y = 2x^3 + 3x^2 - 12x + 6$

g) $y = e^x - 4x$,

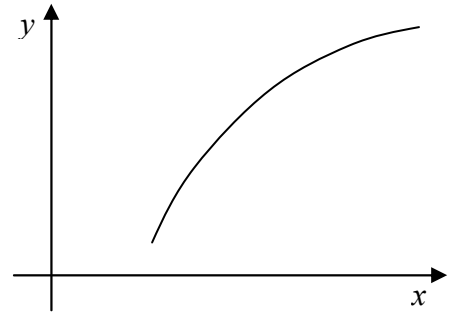
h) $y = e^{2x} - 8x$.

5. For each of the following graphs of $y = f(x)$, write down the signs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

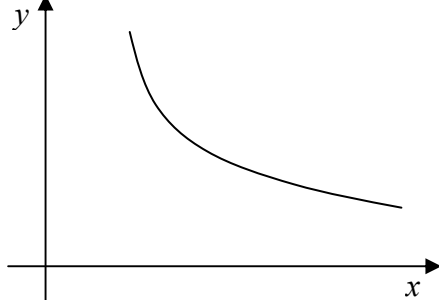
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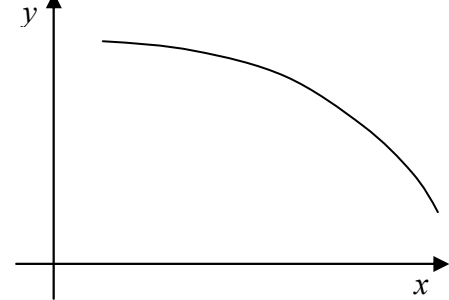
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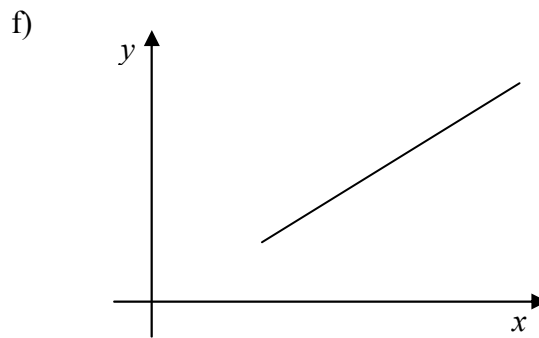
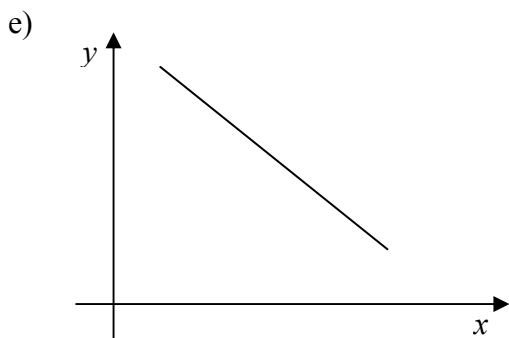


c)



d)





6. Find, by differentiation, **or otherwise**, the co-ordinates of the stationary point on the curve $y = (x - 3)^3$ and hence sketch the graph of $y = (x - 3)^3$.
7. Find the maximum displacement of a particle from a fixed point O , if its displacement s metres from O after time t seconds is given by $s = 2 + 3t - t^2$.
8. Find the co-ordinates and the nature of the turning points on the curve $y = x^3 - 24\ln x$.
9. Find the co-ordinates of the stationary points on the graph of $y = x^3 - 12x - 12$ and sketch the graph.

Find the set of values of k for which the equation $x^3 - 12x - 12 = k$ has more than one real solution.

{Hint: Do NOT attempt to use discriminants! Use the sketch graph etc.}

10. Find the co-ordinates of the stationary points on the curve with equation $y = x(x - 1)^2$. Sketch the curve.

Find the set of real values of k such that the equation $x(x - 1)^2 = k^2$ has exactly one real root.

{Hint: Do NOT attempt to use discriminants! Use the sketch graph etc.}

11. A curve has equation $y = x^2 + \frac{54}{x}$.

Use differentiation to find the co-ordinates of the stationary point and determine, showing your working, whether the stationary point is a maximum point or a minimum point.

Deduce, without differentiation, the co-ordinates of the stationary point of the curve

$$y = (x - 2)^2 + \frac{54}{(x - 2)} - 6.$$

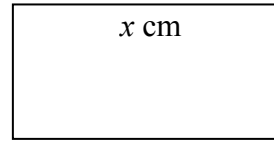
12. The number of bacteria present in a culture at time t hours after the beginning of an experiment is denoted by N . The relation between N and t is modelled by

$$N = 1000 e^{\frac{3}{2}t}.$$

- i) After how many hours will the number of bacteria be 90 000 ?
- ii) At what rate will the number of bacteria be increasing when $t = 6$? Give your answer to the nearest whole number. **{Hint: differentiate.}**

13. A rectangle is to have a **perimeter** of 16cm and a length x cm.

- a) If the length x equals 3cm, find the width and hence the AREA of the rectangle in cm^2 .



- b) Show that the width of the rectangle whose length is x cm, is given by $(8 - x)$ cm.

Hence find a formula, in terms of x , for the area A of the rectangle.

- c) Find $\frac{dA}{dx}$ and hence find the length of the rectangle giving the maximum area.

14. The costs of a firm which makes climbing boots are of two kinds.

Fixed costs (plant, rates, office expenses): £2000 per week;

Production costs (materials, labour): £20 for each pair of boots made.

Market research suggests that, if they price the boots at £30 a pair they will sell 500 pairs a week, but that at £55 a pair they will sell none at all; and between these values the graph of sales against price is a straight line.

If they price boots at £ x a pair ($30 \leq x \leq 55$) find expressions for

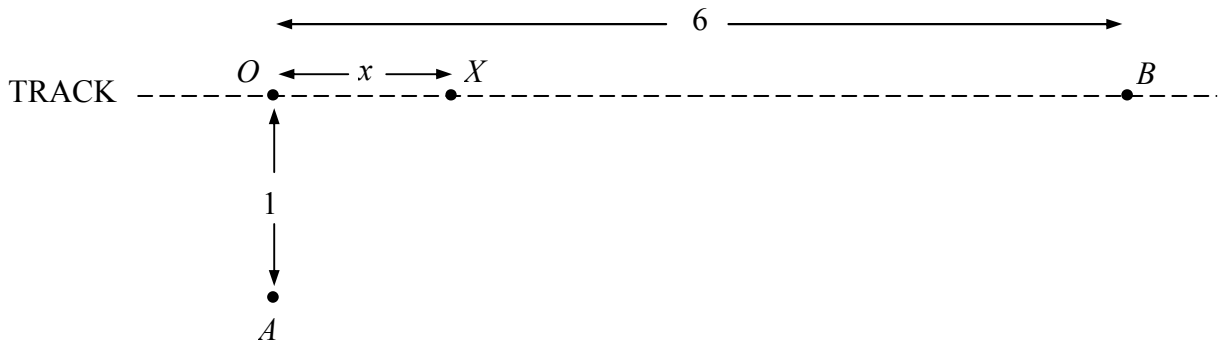
- the number of pairs of boots sold per week
- the weekly receipts,
- the weekly costs

(assuming that just enough boots are made).

Hence show that the weekly profit, £ P , is given by $P = -20x^2 + 1500x - 24000$.

Find the price at which the boots should be sold to maximise the profit.

15.



An orienteer is at a point A , 1 mile across very rough ground from a straight track. The nearest point to A on the track is O . The orienteer needs to reach point B which is 6 miles from O along the track (see diagram). In a simple model of the situation, it is assumed that she can travel at 1 mile per hour across the rough ground and at 6 miles per hour along the track.

Show that the time, in hours, taken to travel from A to B if she first heads directly for X (at a distance x miles from O), and then travels along the track from X to B , is given by

$$1 + \sqrt{1 + x^2} - \frac{1}{6}x.$$

Find the position of X if the time taken to travel from A to B is to be a minimum. (You need not show that this time is a minimum rather than a maximum.)

16. The length of the side of a square is increasing at a constant rate of 1.5 cm s^{-1} . At the moment when the length of the side is 12 cm , find
- the rate of increase of the perimeter,
 - the rate of increase of the area.
17. Ink is leaking at a constant rate of 8 cm^2 per second onto a flat table and forms an ever increasing circle. Find the rate at which the radius of the circle is increasing at the instant when the radius of the circle is 6 cm .
18. Ink is leaking onto a flat table and forms an ever increasing circle in such a way that its radius is increasing at the constant rate of 5 mm s^{-1} . Find the rate at which the area of the circle is increasing at the instant when the radius of the circle is 25 mm .
19. A water tank has a rectangular base 2 m by 1.25 m . The sides are vertical and water is being added to the tank at a constant rate of 0.5 m^3 per minute.
- At what rate is the depth of water in the tank increasing?
(It will be a constant increase if you think about it!)
20. The surface area of a sphere is increasing at a constant rate of $12 \text{ cm}^2 \text{ s}^{-1}$. Find the rate at which its volume is increasing at the moment when the radius is 4 cm .
21. The volume of a sphere is increasing at a constant rate of $10 \text{ cm}^3 \text{ s}^{-1}$. Find the rate at which its surface area is increasing at the moment when the radius is 5 cm .
22. Air is being lost from a spherical balloon at a constant rate of $0.5 \text{ m}^3 \text{ s}^{-1}$. Find the rate at which the radius is decreasing at the instant when the radius is 5 m .
- {Hint: you are given that $\frac{dV}{dt} = -0.5$ etc.}

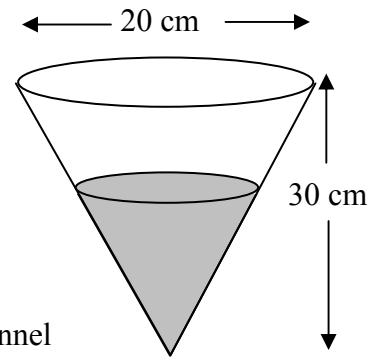
23. A funnel has a circular top of diameter 20 cm and a height of 30 cm.

When the depth of liquid in the funnel is 15 cm, the liquid is dripping from the funnel at a rate of $0.2 \text{ cm}^3 \text{ s}^{-1}$.

At what rate is the depth of the liquid in the funnel **decreasing** at this instant?

{**Hint:** volume of a cone, $V = \frac{1}{3}\pi r^2 h$.

at time t seconds, let $V =$ the volume remaining in the funnel
 $h =$ the depth of the remaining liquid
 and $r =$ the radius of the remaining circle.



To summarise the given information:

when $h = 15$, $\frac{dV}{dt} = -0.2$, find the value of $\frac{dh}{dt}$.

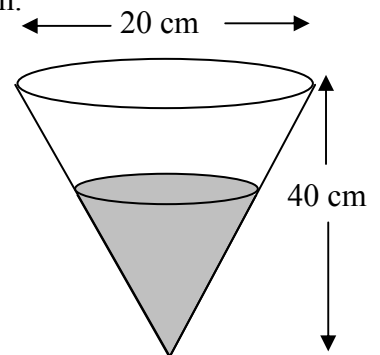
The problem is that your formula for V involves 2 variables, r and h and consequently you are unable to differentiate. You must *remove* one of these quantities. In fact the question makes it clear that you must remove the radius r (why?).

If you look carefully at the diagram you should see that $\frac{r}{h} = \frac{1}{3}$ and hence that $r = \frac{h}{3}$
 etc. (Similar triangles!)

24. A funnel has a circular top of diameter 20 cm and a height of 40 cm.

When the radius of liquid in the funnel is 4 cm, the liquid is dripping from the funnel at a rate of $0.6 \text{ cm}^3 \text{ s}^{-1}$.

At what rate is the radius of the liquid in the funnel **decreasing** at this instant?



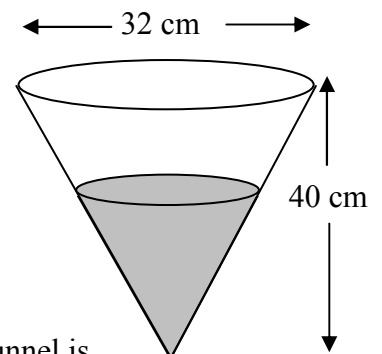
25. A funnel has a circular top of diameter 32 cm and a height of 40 cm.

When the depth of liquid in the funnel is 12 cm, liquid is dripping **into** the funnel at a rate of $0.4 \text{ cm}^3 \text{ s}^{-1}$.

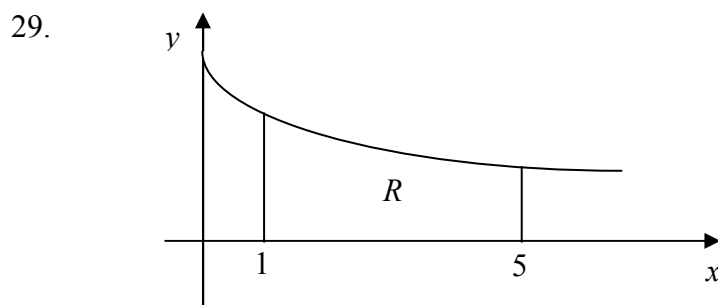
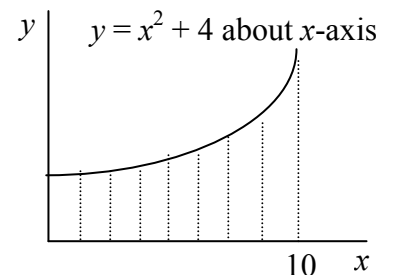
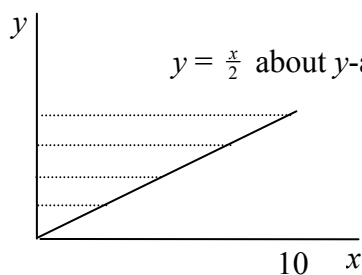
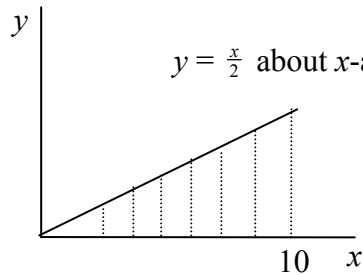
i) Show that the volume, V , of the liquid in the funnel at a given instant can be written in the form $V = \frac{4\pi}{75} h^3$ where h is the depth of liquid at the same instant.

ii) Hence find the rate at which the depth of the liquid in the funnel is **increasing** at the instant when the depth is 12 cm.

iii) At what rate is the **radius** of the liquid in the funnel **increasing** at the instant when the depth is 12 cm?

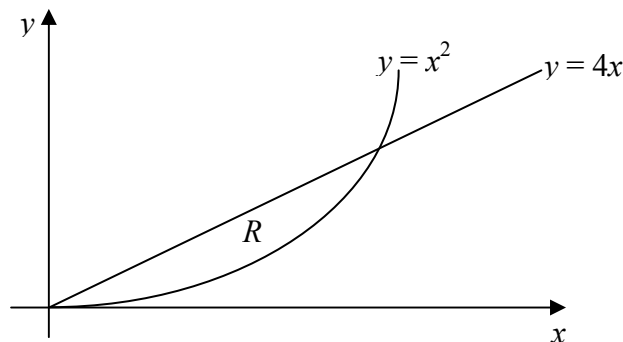


26. Find the volume generated when the region bounded by $y = x^3$ and the x axis between $x = 1$ and $x = 2$ is rotated through 360° about the x -axis.
27. Find the volume generated when the region bounded by $y = x^3$ and the y axis between $y = 1$ and $y = 8$ is rotated through 360° about the y -axis.
 {Hint: $V = \int_1^8 \pi x^2 dy$ and $x = y^{\frac{1}{3}}$ etc.}
28. Find the volumes of the solids formed when the following areas are rotated through 360° about the specified axis.



The diagram shows the region R which is bounded by the curve $y = \frac{2}{x+1}$, the x -axis, and the lines $x = 1$ and $x = 5$. Use integration

- to find the area of R , giving your answers as a single logarithm,
 - to show that the volume of the solid formed when R is rotated completely about the x -axis is $\frac{4}{3}\pi$.
30. The diagram shows the graphs of $y = 4x$ and $y = x^2$.

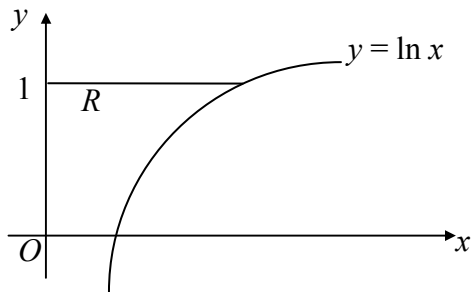


The region enclosed between the 2 graphs is denoted by R . Find the volume generated when R is rotated through 360° about the x -axis.

{Hint: first find the x -coordinates of the points of intersection.}

31. The region R is bounded by the part of the curve $y = (x - 2)^{\frac{3}{2}}$ for which $2 \leq x \leq 4$, the x -axis, and the line $x = 4$. Find, in terms of π , the volume of the solid obtained when R is rotated through 4 right-angles about the x -axis.

32.



The region R , shaded in the diagram, is rotated completely about the **y -axis**. Show that the volume of the solid formed is $\frac{1}{2} \pi(e^2 - 1)$.

{Hint: $V = \int_0^1 \pi x^2 dy$ and $x = e^y$ etc. }

Answers.

1. a) -540 b) -1 c) $9e^4$ d) $-\frac{1}{32}$.

2. $\frac{dy}{dx} = \frac{3}{4 + 3x}$.

4. a) $(0, 0)$ min, $(-\frac{2}{3}, \frac{4}{27})$ max b) $(-1, 4)$ max, $(1, 0)$ min
c) $(-1, -2)$ max, $(1, 2)$ min d) $(2, 2 - 2\ln 2)$ min
e) $(3, 27)$ min f) $(-2, 26)$ max, $(1, -1)$ min
g) $(\ln 4, 4 - 4\ln 4)$ min h) $(\ln 2, 4 - 8\ln 2)$ min.

5. a) both positive b) positive, negative respectively
c) negative, positive respectively d) both negative
e) negative, zero respectively f) positive, zero respectively.

6. $(3, 0)$.

7. 4.25 metres.

8. $(2, -8.636)$ min.

9. $(-2, 4)$, $(2, -28)$, $-28 \leq k \leq 4$.

10. $(\frac{1}{3}, \frac{4}{27})$, $(1, 0)$, $k < -\frac{2}{\sqrt{27}}$ or $k > \frac{2}{\sqrt{27}}$.

11. $(3, 27)$ is a MIN point. $(5, 21)$ is a MIN point on the curve $y = (x-2)^2 + \frac{54}{(x-2)} - 6$.

12. i) 3.00 hours (2 decimal places) ii) 12154625 (nearest integer).

13. a) width = 5 cm, area = 15 cm^2 b) $A = 8x - x^2$

c) $\frac{dA}{dx} = 8 - 2x$, length = 4 cm.

14. a) $\pounds 1100 - 20x$ b) $\pounds x(1100 - 20x)$

c) $\pounds(24000 - 400x)$; $\pounds 37.50$.

15. $x = \frac{1}{\sqrt{35}}$.

16. a) 6 cm s^{-1} b) $36 \text{ cm}^2 \text{ s}^{-1}$.

17. 0.212 cm s^{-1} (3 decimal places).

18. $785.4 \text{ mm}^2 \text{ s}^{-1}$ (1 decimal place).

19. 0.2 m per minute.

20. $24 \text{ cm}^3 \text{ s}^{-1}$.

21. $4 \text{ cm}^2 \text{ s}^{-1}$.

22. 0.00159 m s^{-1} .

23. $0.00255 \text{ cm s}^{-1}$.

24. $0.00298 \text{ cm s}^{-1}$.

25. ii) $0.00552 \text{ cm s}^{-1}$ iii) $0.00221 \text{ cm s}^{-1}$.

26. $\frac{127}{7} \pi \text{ units}^3$.

27. $\frac{93}{5} \pi \text{ units}^3$.

{You didn't expect these last 2 answers to be the same did you?}

28. $\frac{250}{3} \pi \text{ units}^3$, $\frac{500}{3} \pi \text{ units}^3$, $\frac{68480}{3} \pi \text{ units}^3$.

29. i) $\ln 9$.

30. $\frac{2048}{15} \pi \text{ units}^3$.

31. $4\pi \text{ units}^3$.