Logarithmic & Exponential functions

1) Evaluate the following **without a calculator**.

a) $\log_4 64$, b) $\log 10\,000$, c) $\log_3 81$, d) $\log_5 1$, e) $\log_4 2$,

f) $\log_8 2$, g) $\log_2\left(\frac{1}{4}\right)$, h) $\log_2 2^3$, i) $\log_a a^b$.

2) Evaluate the following **without a calculator**.

a) $\log_{\left(\frac{1}{2}\right)}\left(\frac{1}{4}\right)$, b) $\log_{\left(\frac{1}{2}\right)} 2$, c) $\log_{\left(\frac{1}{2}\right)} 4$, d) $\log_{\left(\frac{1}{2}\right)} 8$, e) $\log_{\left(\frac{1}{2}\right)} 1$,

f) $\log_{\left(\frac{2}{3}\right)}\left(\frac{8}{27}\right)$, g) $\log_{\left(\frac{2}{3}\right)}\left(\frac{27}{8}\right)$, h) $\log_{\left(\frac{3}{4}\right)}\left(\frac{16}{9}\right)$.

3) Solve the equation $10^x = 120$.

*4) Solve the equation $10^y = 200$.
Hence solve the equation $100^x = 200$.

5) Simplify the following **without a calculator**.

a) $\log 5 + \log 20$,
b) $\log_a 7 + 2\log_a 3 - 3\log_a 5$,
c) $3\log 5 + 4\log 2$,
d) $\log_a 9 + 2\log_a 2 - 3\log_a 3$,
e) $5\log 3 - 2\log 5$,
f) $\log \sqrt{10}$,
g) $\log \sqrt[3]{\frac{1}{1000}}$,
h) $\log_2 p + 2\log_2 q$,
i) $\log_3 r - 2\log_3 s$,
j) $2\log x + \log y - \log xy$,
k) $\log(x^2 - 1) - \log(x + 1)$,
l) $\log(x^2 + 4x) - \log x$.

6) Solve the following equations by using the LOG key on your calculator.

a) $10^x = 150$, b) $10^{-x} = 0{\cdot}2$, c) $3^x = 20$, d) $5^x = 10$, e) $5^{2x+1} = 100$.

7) Solve the following equations.

a) $\log_2 x = 6$, b) $\log_3 x = 2{\cdot}5$, c) $\log 5x = 1{\cdot}5$, d) $\log(3x + 1) = 0$.

8) Express $\log_2(x + 2) - \log_2 x$ as a single logarithm.

Hence solve the equation $\log_2(x + 2) - \log_2 x = 3$.

9) Show, by taking log's, that the equation $2^{3x} \times 3^{2x} = 100$, is equivalent to $x(3\log 2 + 2\log 3) = 2$.

Hence find the value of x correct to 3 significant figures.

10) Sketch the graph of $y = e^x$.

Describe the transformation which maps the graph of $y = e^x$ onto that of $y = e^{-x}$ and thus sketch the graph of $y = e^{-x}$.

11) The strength of a radioactive source is said to 'decay exponentially'. Explain briefly what is meant by exponential decay, and illustrate your answer by means of a sketch–graph.

After t years the strength s of a particular radioactive source, in appropriate units, is given by $s = 15000e^{-0{\cdot}002t}$.

State the value of s when $t = 0$, and find the value of t when the source has decayed to one-half of its initial strength, giving your answer correct to 3 significant figures.

12) The number of bacteria present in a culture at time t hours after the beginning of an experiment is denoted by N. The relation between N and t is modelled by

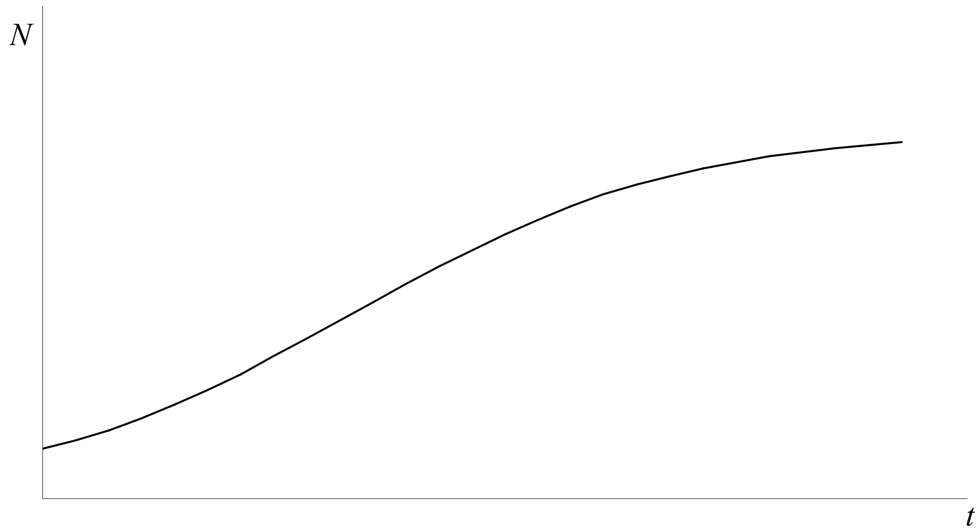
$$N = 100e^{\frac{2}{3}t}.$$

After how many hours will the number of bacteria be 9000?

13) The number of insects of a certain species was recorded at a particular site at weekly intervals. The number of insects present after t weeks is denoted by N. A model for the number of insects present is

$$N = \frac{150}{2 + 13e^{-0{\cdot}4t}}.$$

The graph of this relation is shown.



i) State the feature of the graph which indicates that the population growth is not exponential.
ii) According to the model, how many insects were present after six weeks?
iii) According to the model, after how many weeks were 67 insects present?
iv) As t increases, N approaches a certain value. State that value.

14) Solve the equation $8e^{x+1} = 9$, leaving your answer in terms of ln 2 and ln 3.

15) Solve the equation $16e^{3x-1} = 27$, giving your answer in terms of ln 2 and ln 3.

16) Given that $\ln p = 1600$, find the values of i) $\ln(pe^6)$, ii) $\ln\left(\frac{1}{\sqrt{p}}\right)$.

17) i) Solve the equation $\ln\left(\frac{1}{2}x + 1\right) = 8$, giving your answer in terms of e.

ii) Solve the equation $e^{\frac{1}{2}x + 1} = 8$, giving your answer in terms of ln 2.

18) Find y in terms of x for each of the following equations:

i) $3^y = 5^x$, expressing your answer in the form $y = cx$ and giving the exact value of the constant c.

ii) $\ln(x^4) - \ln(x^2y) + \ln(y^2) = 0$, expressing your answer in the form $y = kx^n$ and giving the values of the constants k and n.

*19) A sequence $u_1, u_2, u_3, \ldots\ldots$ is defined by $u_n = 8\left(\left(\frac{5}{4}\right)^n - 1\right)$ for all positive integers n.

i) Describe the behaviour of the sequence as n increases.

ii) Show that $u_{n+1} - u_n = \dfrac{5^n}{2^{2n-1}}$.

iii) Given that $P_n = (u_{n+1} - u_n)(u_n - u_{n-1})(u_{n-1} - u_{n-2})\ldots(u_3 - u_2)(u_2 - u_1)$,

show that $\ln P_n = \dfrac{1}{2}n(n+1)\ln 5 - n^2\ln 2$.

Hence show that $P_{20} \approx 2{\cdot}4\times10^{26}$.

ANSWERS.

1) a) 3, b) 4, c) 4, d) 0, e) $\frac{1}{2}$, f) $\frac{1}{3}$, g) −2, h) 3, i) b.

2) a) 2, b) −1, c) −2, d) −3, e) 0, f) 3, g) −3, h) −2.

3) $x = \log 120 = 2{\cdot}079181246$.

4) $y = \log 200 = 2{\cdot}301029996$, $x = 1{\cdot}150514998$.

5) a) 2, b) $\log_a\left(\frac{63}{125}\right)$, c) $\log 2000$ or $(3 + \log 2)$, d) $\log_a\left(\frac{4}{3}\right)$, e) $\log\left(\frac{243}{25}\right)$, f) $\frac{1}{2}$, g) −1, h) $\log_2 pq^2$,
i) $\log_3\left(\frac{r}{s^2}\right)$, j) $\log x$, k) $\log(x - 1)$, l) $\log(x + 4)$.

6) a) $x = 2{\cdot}176091259$, b) $x = 0{\cdot}698970004$, c) $x = 2{\cdot}726833028$, d) $x = 1{\cdot}430676558$, e) $x = 0{\cdot}930676558$.

7) a) $x = 64$, b) $x = 15{\cdot}58845727$, c) $x = 6{\cdot}32455532$, d) $x = 0$.

8) $\log_2\left(\frac{x+2}{x}\right)$, $x = \frac{2}{7}$.

9) $x = 1{\cdot}08$ (3 significant figures).

10) A reflection in the y–axis.

11) When $t = 0$, $s = 15000$, 347 years (3 sig. figs.).

12) 6·7 hours.

13) i) The graph exhibits a point of inflexion which indicates that the growth is not exponential, ii) 47,
iii) 10, iv) 75.

14) $x = 2\ln 3 - 3\ln 2 - 1$.

15) $x = \frac{1}{3}\{3\ln 3 - 4\ln 2 + 1\}$.

16) i) 1606, ii) −800.

17) i) $x = 2e^8 - 2$, ii) $x = 6\ln 2 - 2$.

18) i) $y = \left(\frac{\ln 5}{\ln 3}\right)x$, $c = \left(\frac{\ln 5}{\ln 3}\right)$, ii) $y = x^{-2}$, $k = 1$, $n = -2$.