# FURTHER DIFFERENTIATION.

Remember, that unless otherwise instructed, you are free to use your formula books to assist with any working.

1) Show that $\frac{d}{dx}(\operatorname{cosech} x) = -\coth x \operatorname{cosech} x$.

2) Show that $\frac{d}{dx}(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$.

3) Show that $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x$.

4) Use the method of implicit differentiation to obtain the results in the table below.

| $f\ (x)$ | $f'(x)$ |
|---|---|
| $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ |
| $\cos^{-1} x$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\tan^{-1} x$ | $\frac{1}{1+x^2}$ |
| $\sinh^{-1} x$ | $\frac{1}{\sqrt{1+x^2}}$ |
| $\cosh^{-1} x$ | $\frac{1}{\sqrt{x^2-1}}$ |
| $\tanh^{-1} x$ | $\frac{1}{1-x^2}$ |

5) Differentiate the following with respect to $x$.

a) $y = \cosh^{-1} x$, b) $y = \sin^{-1}(2x)$, c) $y = \tan^{-1}(x+1)$,
d) $y = \sin^{-1}(2x+1)$, e) $y = \cosh^{-1}(x^2)$, f) $y = \tan^{-1}\left(\sqrt{x}\right)$.

6) Given that $y = \sin^{-1}(2x-1)$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{x - x^2}}$.

7) Given that $y = \cos^{-1}(e^x)$, show that $\frac{dy}{dx} = -\frac{e^x}{\sqrt{1 - e^{2x}}}$.

8) Given that $y = \cos^{-1}(\sin 2x)$, show that $\frac{dy}{dx} = -2$. {This means that that $y$ is a straight line!}

9) Show that $\tanh y = \frac{e^{2y} - 1}{e^{2y} + 1}$.

Hence show that $\tanh^{-1} x = \frac{1}{2}\ln(1 + x) - \frac{1}{2}\ln(1 - x)$ and deduce that $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1 - x^2}$.

10) If $x^2 + y^2 = 2y$, find $\frac{dy}{dx}$ in terms of $x$ and $y$ without first finding $y$ in terms of $x$.

Prove that $\frac{d^2y}{dx^2} = \frac{1}{(1 - y)^3}$.

11) Given that $x^3 + y^3 = 1$,

i) find an expression for $\frac{dy}{dx}$ in terms of $x$ and $y$,

ii) show that $\frac{d^2y}{dx^2} = \frac{-2x}{y^5}$.

12) Find $\frac{d}{dx}\left(x \sinh^{-1} x\right)$.

{**Hint:** use the fact that $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1 + x^2}}$, which is in your formula book.}

13) Given that $y = \ln(1 + \sin x)$, show that $\frac{dy}{dx} = \frac{\cos x}{e^y}$. Deduce that $\frac{d^2y}{dx^2} + e^{-y} = 0$.

14) Given that $y = \ln(1 - \cos x)$, show that $\frac{dy}{dx} = \frac{\sin x}{e^y}$. Deduce that $\frac{d^2y}{dx^2} + e^{-y} = 0$.

15) The parametric equations of a curve are $x = e^{2t} - 5t$, $y = e^{2t} - 2t$. Find $\frac{dy}{dx}$ in terms of $t$.

Find the exact value of $t$ at the point on the curve where the gradient is 2.

16) A curve has parametric equations $x = 2t - \ln(2t)$, $y = t^2 - \ln(t^2)$, where $t > 0$.

Find the value of $t$ at the point on the curve at which the gradient is 2.

**You might find the double-angle formulae useful for the following questions.**

17) The parametric equations of a curve are $x = 2\cos t$, $y = 5 + 3\cos 2t$, where $0 < t < \pi$.

Express $\frac{dy}{dx}$ in terms of $t$, simplifying your answer, and hence show that the gradient at any point on the curve is less than 6.

18) A curve has parametric equations $x = 2t + \sin 2t$, $y = \cos 2t$, where $0 < t < \frac{1}{2}\pi$.

Show that, at the point with parameter $t$, the gradient of the curve is $-\tan t$.

19) A curve has parametric equations $x = a(t - \sin t)$, $y = a(1 - \cos t)$, show that $\frac{dy}{dx} = \cot \frac{1}{2}t$.

20) a) Show that $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$.

b) A curve $C$ has parametric equations $x = t - \tanh t$, $y = \frac{1}{\cosh t}$.

Show that $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \tanh^2 t$.

{**Hint:** you might need to use the identity $1 - \tanh^2 t \equiv \operatorname{sech}^2 t$, though some methods will avoid this!}

ANSWERS. {Where appropriate, answers given are simplified as far as possible.}

5) a) $\dfrac{1}{\sqrt{x^2-1}}$, b) $\dfrac{2}{\sqrt{1-4x^2}}$, c) $\dfrac{1}{1+(x+1)^2}$, d) $\dfrac{2}{\sqrt{1-(2x+1)^2}}$ or $\dfrac{1}{\sqrt{-x^2-x}}$, e) $\dfrac{2x}{\sqrt{x^4-1}}$,

f) $\dfrac{1}{2\sqrt{x}(1+x)}$.

10) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x}{1-y}$.

11) i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{x^2}{y^2}$.

12) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\sinh^{-1}x+\dfrac{x}{\sqrt{1+x^2}}$.

15) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2e^{2t}-2}{2e^{2t}-5}$. $t=\ln 2$.

16) $t=2$.

17) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3\sin 2t}{\sin t}$ which simplifies to $6\cos t$.