BASIC COMPLEX NUMBERS

1) Express the following as complex numbers.

a) $(3 + 5i) + (7 + 2i)$, b) $(3 + 2i) + (1 - 2i)$, c) $(3 - 2i) + (-3 - 2i)$,
d) $(3 + 5i) - (7 + 2i)$, e) $(3 + 2i) - (1 - 2i)$, f) $(3 - 2i) - (-3 - 2i)$,
g) $(3 + 5i) \times (7 + 2i)$, h) $(3 + 2i) \times (1 - 2i)$, i) $(3 - 2i) \times (3 + 2i)$.

2) Divide the following by multiplying throughout by the complex conjugate of the denominator. Express each answer in the form $a + b$i.

a) $\dfrac{3 + i}{4 + 3i}$, b) $\dfrac{3 + i}{4 - 3i}$, c) $\dfrac{3}{2 + i}$, d) $\dfrac{4i}{4 + i}$,
e) $\dfrac{2 - 2i}{3 - 2i}$.

3) For each of the following, solve to find the real numbers x and y.

a) $3 + 5i + x - yi = 6 - 2i$, b) $x + yi = (1 - i)(2 + 8i)$,
c) $x + yi = \dfrac{2 + 5i}{1 - i}$, d) $x + yi = (4 + i)^2$.

4) i) Find the value of the real number y such that $(3 + 2i) \times (1 + yi)$ is real.
ii) Find the value of the real number y such that $(3 + 2i) \times (1 + yi)$ is imaginary.

5) Find the square-roots of the following.

a) $5 + 12i$, b) $3 - 4i$, c) $21 - 20i$, d) $2i$.

6) Solve the following equations.

a) $x^2 + 6x + 10 = 0$, b) $x^2 + x + 1 = 0$, c) $2x^2 + 7x + 1 = 0$,
d) $x^2 + 9 = 0$, e) $x^2 + x + 3 = 0$, f) $x^4 - 1 = 0$ {**Hint:** factorise.}

7) One root of a quadratic equation with real coefficients is $z = 1 + 2i$, find the equation.

8) One root of a quadratic equation with real coefficients is $z = 2 + i$, find the equation.

9) One root of a quadratic equation with real coefficients is $z = 1 - 3i$, find the equation.

10) Given that $x = 2$ is a solution of the equation $x^3 - 4x^2 + 6x - 4 = 0$, find the other two roots.

11) Given that $x = 1$ is a solution of the equation $x^3 - 7x^2 + 19x - 13 = 0$, find the other two roots.

12) The (complex) equation $z^2 + (a + bi)z - (9 + 23i) = 0$ has a root $3 + i$. Find the real numbers a and b.

13) The (complex) equation $z^2 + (a + bi)z - 5(1 + 5i) = 0$ has a root $3 + 2i$. Find the real numbers a and b.

*14) The equation $x^3 - 3x^2 + x + 5 = 0$ has a root $2 + i$. Find the other two roots.

{**Hint:** i) $z = 2 + i$ being a root $\Rightarrow z^* = 2 - i$ is a root,
ii) find a quadratic equation with roots $2 + i$ and $2 - i$ as in Q7,
iii) now use long division to find the 3rd root!}

*15) The equation $z^3 - 11z + 20 = 0$ has a root $2 - i$. Find the other two roots.

16) Represent the following complex numbers by lines on Argand diagrams. Determine the modulus and argument of each complex number.

a) $3 - 2i$, b) $-4 + i$, c) $-3 - 4i$, d) $5 + 12i$, e) $1 - i$, f) $-1 + i$,
g) 4, h) $-2i$, i) $1 + i$.

17) If $z_1 = 3 - i$, $z_2 = 1 + 4i$, $z_3 = -4 + i$, $z_4 = -2 - 5i$, represent the following by lines on Argand diagrams, showing the direction of each line by an arrow.

a) $z_1 + z_2$, b) $z_2 - z_3$, c) $z_1 - z_3$, d) $z_2 + z_4$, e) $z_4 - z_1$, f) $z_3 - z_4$.

18) For each complex number in 1) a), b) and c), find the modulus and argument of the <u>conjugate</u> complex number. **{Draw Argand diagrams!}**
What conclusion can you reach concerning the moduli and arguments of conjugate complex numbers?

19) Express the following complex numbers in the form $r(\cos\theta + i\sin\theta)$.

a) $1 + i$, b) $\sqrt{3} - i$, c) $-3 - 4i$, d) $-5 + 12i$, e) $2 - i$,
f) 6, g) -3, h) $4i$, i) $-3 - \sqrt{3}i$, j) $24 + 7i$.

20) Express the following in the form $z = a + bi$.

a) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, b) $3\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$, c) $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$,
d) $3(\cos 0 + i\sin 0)$, e) $2(\cos\pi + i\sin\pi)$, f) $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

21) If $z_1 = 2\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$ and $z_2 = 6\left[\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right]$, find

a) $\left|\frac{z_1}{z_2}\right|$, b) $\arg\left(\frac{z_1}{z_2}\right)$, c) $\left|\frac{z_2}{z_1}\right|$, d) $\arg\left(\frac{z_2}{z_1}\right)$

22) The complex numbers z and w are given by $z = 2 + \sqrt{3}i$ and $w = 2 - \sqrt{3}i$.

i) Represent z and w on an Argand diagram.

ii) Find $|z|$ and $|w|$ and hence find $\left|\frac{z}{w}\right|$.

iii) Find arg (z) and arg (w) and hence find $\arg\left(\frac{z}{w}\right)$.

Hence express the complex number $\dfrac{2 + \sqrt{3}i}{2 - \sqrt{3}i}$ in modulus, argument form.

23) Given that $z_1 = 1 + i$ and $z_2 = \sqrt{3} + i$, find the following complex numbers in modulus, argument form.

a) z_1, b) z_2, c) $z_1 \times z_2$, d) $\frac{z_1}{z_2}$, e) $\frac{z_2}{z_1}$.

24) Given that $z_1 = 1 + i$ and $z_2 = 1 - \sqrt{3}i$, find the following complex numbers in modulus, argument form.

a) z_1, b) z_2, c) $z_1 \times z_2$, d) $\frac{z_1}{z_2}$, e) $\frac{z_2}{z_1}$, f) z_2^2.

25) If $z_1 = 4\left[\cos\frac{13\pi}{24} + i\sin\frac{13\pi}{24}\right]$ and $z_2 = 2\left[\cos\frac{5\pi}{24} + i\sin\frac{5\pi}{24}\right]$, express the following complex numbers in the form $a + bi$. a) $\frac{z_1}{z_2}$, b) $z_1 \times z_2$.

26) If $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$, draw on an Argand diagram, points representing the complex numbers z, z^2, z^3 and z^4.

27) If $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$, draw on an Argand diagram, points representing the complex numbers z, z^2, z^3 z^4, z^5 and z^6.

28) Given that $(5 + 12i)z = -7 + 17i$, express z in the form $a + bi$ and hence find $|z|$ and arg (z).
Given also that $w = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, find i) $\left|\frac{z}{w}\right|$ and ii) arg (zw).

29) Given that $(5 + 12i)z = -4 + 58i$, find $|z|$ and arg (z), giving your answers to 3 significant figures.
Given also that $w = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, find i) $\left|\frac{z}{w}\right|$ and ii) arg (zw).

ANSWERS.

1) a) $10 + 7i$, b) 4, c) $-4i$, d) $-4 + 3i$, e) $2 + 4i$, f) 6, g) $11 + 41i$, h) $7 - 4i$, i) 13.

2) a) $\left(\frac{3}{5}\right) - \left(\frac{1}{5}\right)i$, b) $\left(\frac{9}{25}\right) + \left(\frac{13}{25}\right)i$, c) $\left(\frac{6}{5}\right) - \left(\frac{3}{5}\right)i$, d) $\left(\frac{4}{17}\right) + \left(\frac{16}{17}\right)i$, e) $\left(\frac{10}{13}\right) - \left(\frac{2}{13}\right)i$.

3) a) $x = 3, y = 7$. b) $x = 10, y = 6$. c) $x = -\frac{3}{2}, y = \frac{7}{2}$, d) $x = 15, y = 8$.

4) i) $y = -\frac{2}{3}$, ii) $y = \frac{3}{2}$.

5) a) $3 + 2i$ or $-3 - 2i$, b) $2 - i$ or $-2 + i$, c) $5 - 2i$ or $-5 + 2i$, d) $1 + i$ or $-1 - i$.

6) a) $x = -3 \pm i$, b) $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}\,i$, c) $x = -\frac{7}{4} \pm \frac{1}{4}\sqrt{41}\,i$, d) $x = \pm 3i$, e) $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{11}\,i$,
f) $x = -1, 1, i, -i$.

7) $z^2 - 2z + 5 = 0$.

8) $z^2 - 4z + 5 = 0$.

9) $z^2 - 2z + 10 = 0$.

10) $x = 1 + i$ or $x = 1 - i$.

11) $x = 3 + 2i$ or $x = 3 - 2i$.

12) $a = 2, b = 5$.

13) $a = 2, b = 3$.

14) Roots $2 + i, 2 - i, -1$.

15) Roots $2 - i, 2 + i, -4$.

16) a) $r = \sqrt{13}$, $\theta = -0{\cdot}588$, b) $r = \sqrt{17}$, $\theta = 2{\cdot}897$, c) $r = 5$, $\theta = -2.214$, d) $r = 13$, $\theta = 1{\cdot}176$,
e) $r = \sqrt{2}$, $\theta = -\frac{\pi}{4}$, f) $r = \sqrt{2}$, $\theta = \frac{3\pi}{4}$, g) $r = 4$, $\theta = 0$, h) $r = 2$, $\theta = -\frac{\pi}{2}$, i) $r = \sqrt{2}$, $\theta = \frac{\pi}{4}$.

18) a) $r = \sqrt{13}$, $\theta = 0{\cdot}588$, b) $r = \sqrt{17}$, $\theta = -2{\cdot}897$, c) $r = 5$, $\theta = 2{\cdot}214$.

19) a) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, b) $2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$, c) $5[\cos(-2{\cdot}214) + i\sin(-2{\cdot}214)]$,
d) $13(\cos 1{\cdot}966 + i\sin 1{\cdot}966)$, e) $\sqrt{5}\,[\cos(-0{\cdot}464) + i\sin(-0{\cdot}464)]$, f) $6(\cos 0 + i\sin 0)$,
g) $3(\cos\pi + i\sin\pi)$, h) $4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$, i) $2\sqrt{3}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$, j) $25(\cos 0{\cdot}284 + i\sin 0{\cdot}284)$.

20) a) $\sqrt{3} + i$, b) $\frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}}i$, c) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, d) 3, e) -2, f) $2i$.

21) a) $\frac{1}{3}$, b) $\frac{17\pi}{12}$, c) 3, d) $-\frac{17\pi}{12}$.

22) ii) $|z| = \sqrt{7}$, $|w| = \sqrt{7}$, $\left|\frac{z}{w}\right| = 1$. iii) $\arg(z) = 0{\cdot}714$, $\arg(w) = -0{\cdot}714$, $\arg\left(\frac{z}{w}\right) = 1{\cdot}427$.
$\cos(1{\cdot}427) + i\sin(1{\cdot}427)$.

23) a) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, b) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, c) $2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$, d) $\frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$,
e) $\sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$.

24) a) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, b) $2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$, c) $2\sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$,
d) $\frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$, e) $\sqrt{2}\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$, f) $4\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$.

25) a) $1 + \sqrt{3}\,i$, b) $-\frac{8}{\sqrt{2}} + \frac{8}{\sqrt{2}}i$.

28) $z = 1 + i$, $|z| = \sqrt{2}$, $\arg(z) = \frac{\pi}{4}$. i) $\frac{1}{\sqrt{2}}$, ii) $\frac{5\pi}{12}$.

29) $z = 4 + 2i$, $|z| = \sqrt{20}$, $\arg(z) = 0{\cdot}464$. i) $\sqrt{5}$, ii) $0{\cdot}987$.