INTEGRATION

Section A.

Integrate the following functions with respect to x.

1) x^7, 2) $x^{\frac{1}{3}}$, 3) $\sqrt[4]{x}$, 4) $x^{-\frac{3}{2}}$,

5) $\frac{1}{x^5}$, 6) $3x^2 - 4x + 1$, 7) $\sqrt{x} + \frac{1}{\sqrt{x}}$, 8) $2x^2 + x^{-2}$,

9) $5x^4 - 4x^3 + 2x$, 10) $9\sqrt{x}$, 11) $\left(\sqrt{x}\right)^{-5}$, 12) $\frac{1}{2}x - \frac{2}{\sqrt{x}}$,

13) $2x^4 + \sqrt{x}$, 14) $6\sqrt[3]{x}$, 15) $10\sqrt[3]{x^2}$, 16) $(x + 1)^5$,

17) $(2x + 1)^3$, 18) $(3x - 1)^2$, 19) $(5 - 3x)^4$, 20) $2(1 + 3x)^3$,

21) $3(2 - 5x)^4$, 22) $10(2 - 3x)^5$, 23) $8(1 + 5x)^3$, 24) $\sqrt{2 + x}$,

25) $\sqrt{5 - 2x}$, 26) $\sqrt[3]{1 + 3x}$, 27) $\sqrt[3]{3 - 2x}$, 28) $(3x + 2)^{\frac{3}{2}}$,

29) $\frac{1}{\sqrt{2x - 4}}$, 30) $\frac{1}{\sqrt[3]{5x - 1}}$, 31) $\frac{1}{(4x - 1)^3}$, 32) $\frac{3}{(2x + 1)^2}$,

33) $\frac{4}{\sqrt{3x - 1}}$, 34) $\frac{8}{(5 - 2x)^5}$, 35) $\frac{5}{3\sqrt{x + 1}}$, 36) e^{2x},

37) $e^{3x + 1}$, 38) $6e^{3x - 2}$, 39) $e^{2 - 4x}$, 40) $\frac{1}{e^{2x}}$,

41) $3e^{-3x}$, 42) $\frac{10}{e^{5x}}$, 43) $\sqrt{e^x}$, 44) $\frac{1}{3 + x}$,

45) $\frac{2}{1 + 2x}$, 46) $\frac{8}{3 + 2x}$, 47) $\frac{1}{2 + 5x}$, 48) $\frac{3}{1 + 4x}$,

49) $\frac{2}{1 - 2x}$, 50) $\frac{9}{2 - 3x}$, 51) $\frac{3}{2 - 5x}$.

Section B.

1) Find the following definite integrals.

a) $\int_{2}^{3} (x^2 + 2x - 1)\, dx$, b) $\int_{0}^{1} e^{4x}\, dx$, c) $\int_{3}^{4} \frac{1}{x + 1}\, dx$,

d) $\int_{2}^{7} \sqrt{x + 2}\, dx$, e) $\int_{1/4}^{1/2} \frac{1}{x^3}\, dx$, f) $\int_{4}^{9} \frac{1}{\sqrt{x}}\, dx$,

g) $\int_{0}^{4} (x^3 - 2x - 3\sqrt{x})\, dx$.

2) Find the areas bounded by the specified lines and curves. **{Sketch graphs where possible!}**

a) The x–axis, the y–axis, the line $x = 3$ and the curve $y = x^2 + 1$.

b) The x–axis, the lines $x = 1$ and $x = 4$ and the curve $y = \frac{1}{x}$.

c) The x–axis and the curve $y = 4 - x^2$.

d) The x–axis and the curve $y = x(x^2 - 16)$.

e) The x–axis and the curve $y = x^2 - 9$.

f) The x–axis, the lines $x = 0$ and $x = 4$ and the curve $y = e^{2x} + \sqrt{x}$.
{Hint: the curve doesn't cut the x–axis;– why not?}

g) The x–axis and the curve $y = 2 + x - x^2$.

3) Evaluate the definite integral $\int_{2}^{5} (2x + 1) dx$.

Interpret this integral geometrically by considering the area of a suitable trapezium.

4) By considering the area of a suitable trapezium, evaluate, **without integrating**, $\int_{2}^{6} (20 - 2x) dx$.

5) Find the area of the region bounded by the x–axis, the curve $y = (x - 6)^2$ and the lines $x = 2$ and $x = 5$.

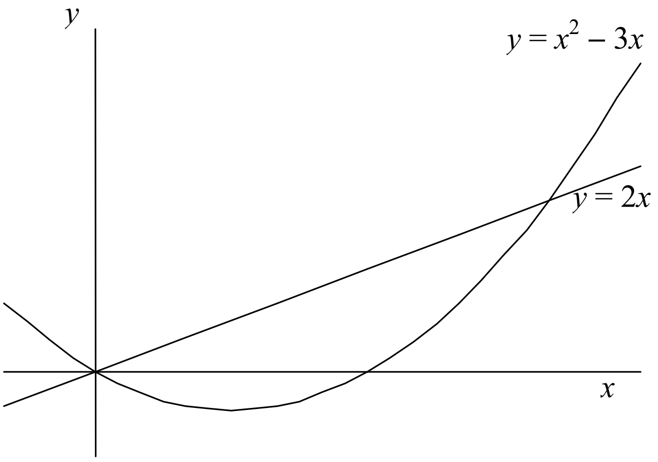
Without further integration, state the value of the definite integral $\int_{2}^{5} \left(12x - x^2 - 36\right) dx$.

6) Evaluate the definite integrals i) $\int_{-1}^{0} x^3 dx$, ii) $\int_{0}^{1} x^3 dx$, and iii) $\int_{-1}^{1} x^3 dx$.

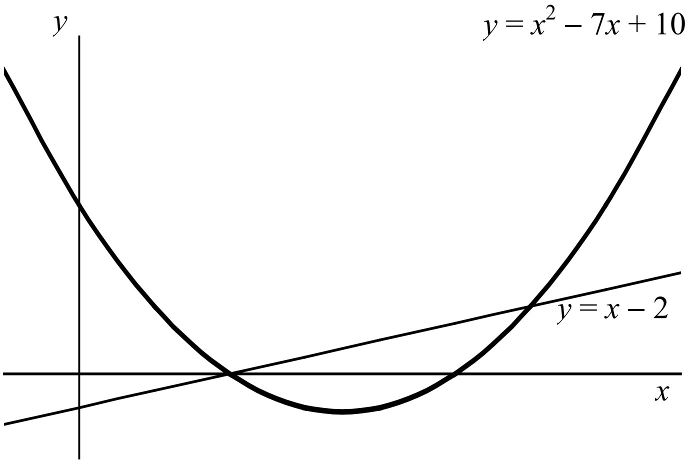
Interpret these results geometrically by sketching graphs.

7) For the following graphs, i) find the co–ordinates of the points of intersection,
and ii) find the area between the graphs.

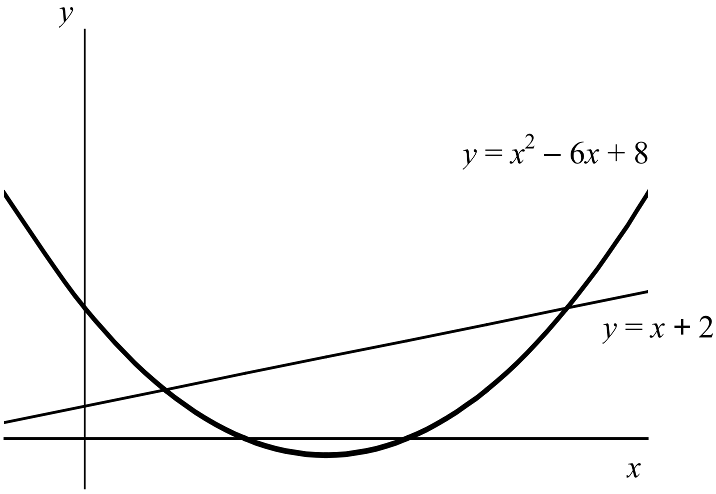
a)



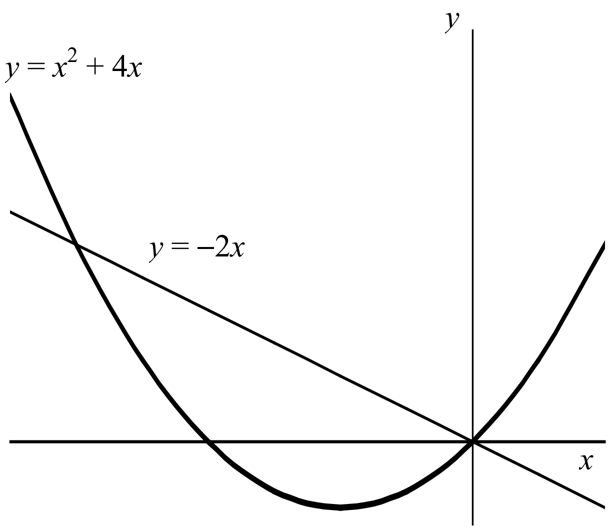
b)



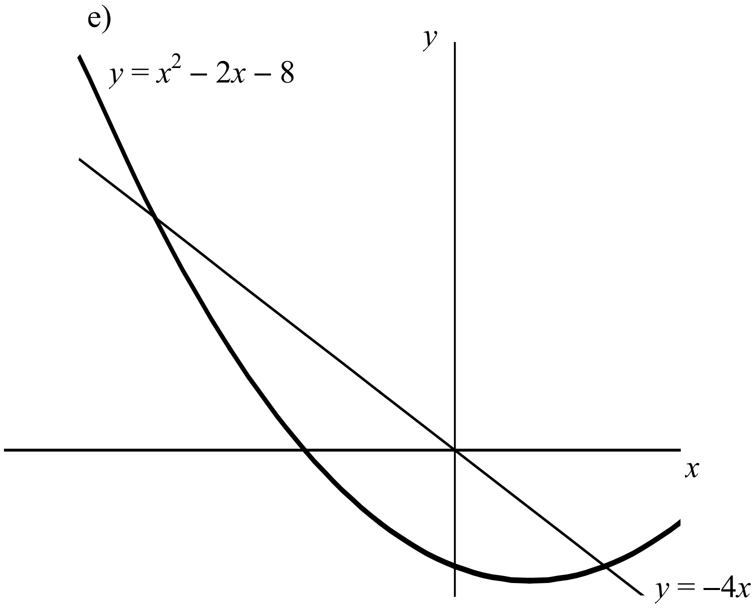
c)



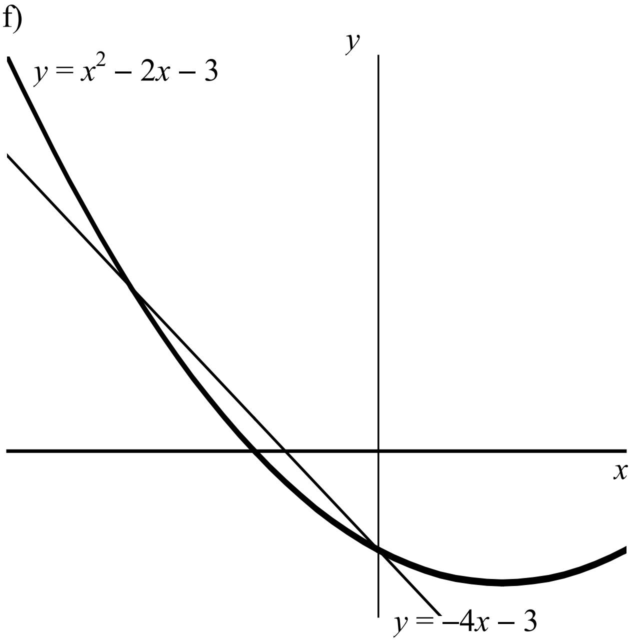
d)



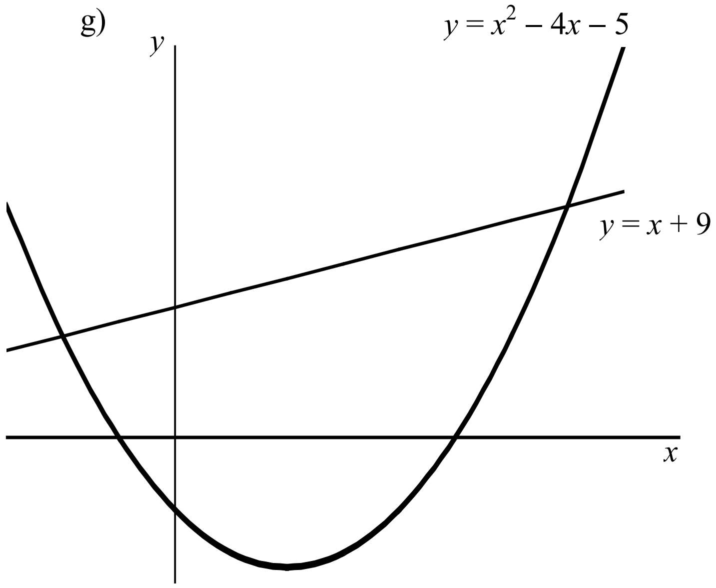
e)



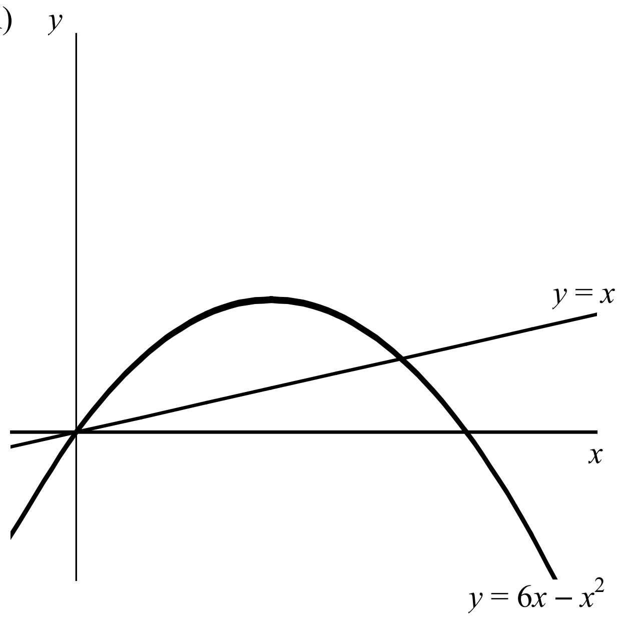
f)



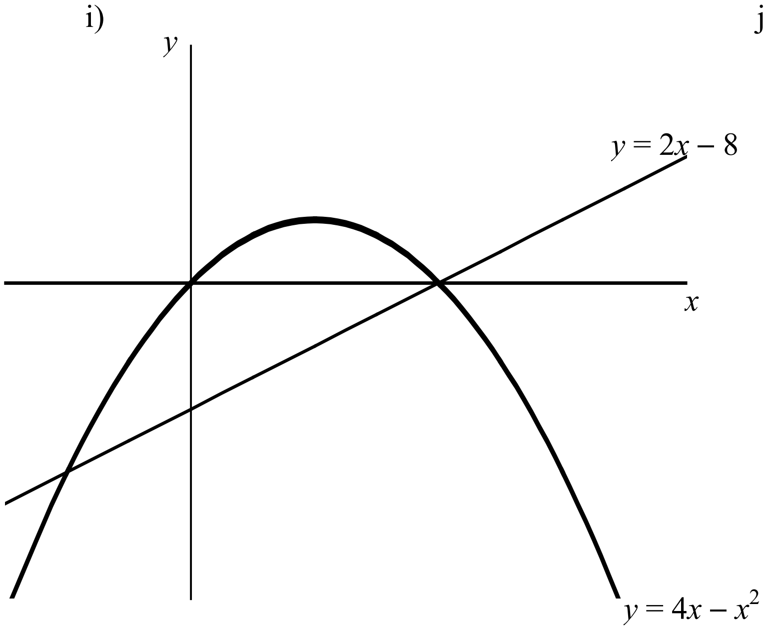
g)


y = x2 − 4x − 5
y
y = x + 9
x

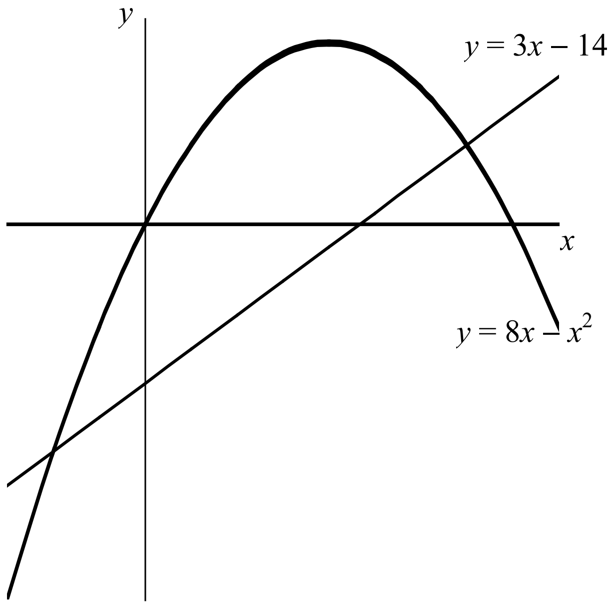
h)


y
y = x
x
y = 6x − x2

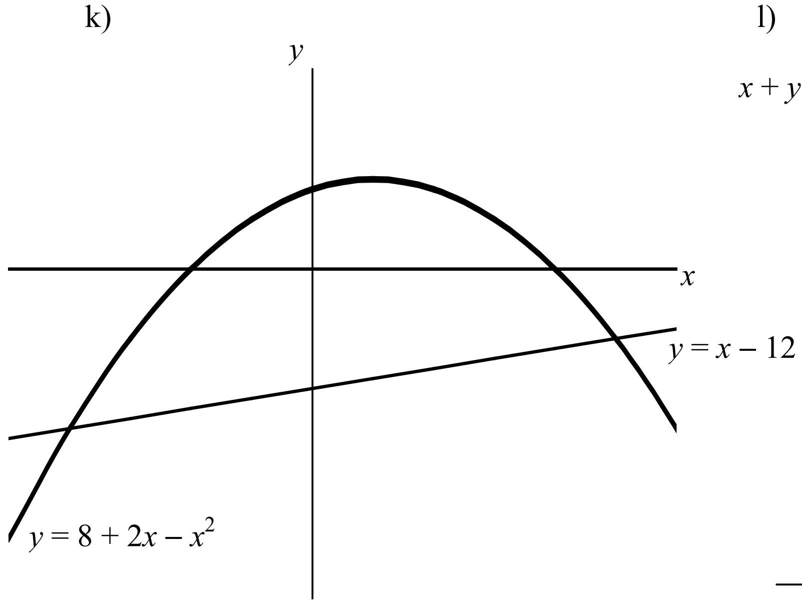
i)


y
y = 2x − 8
x
y = 4x − x2

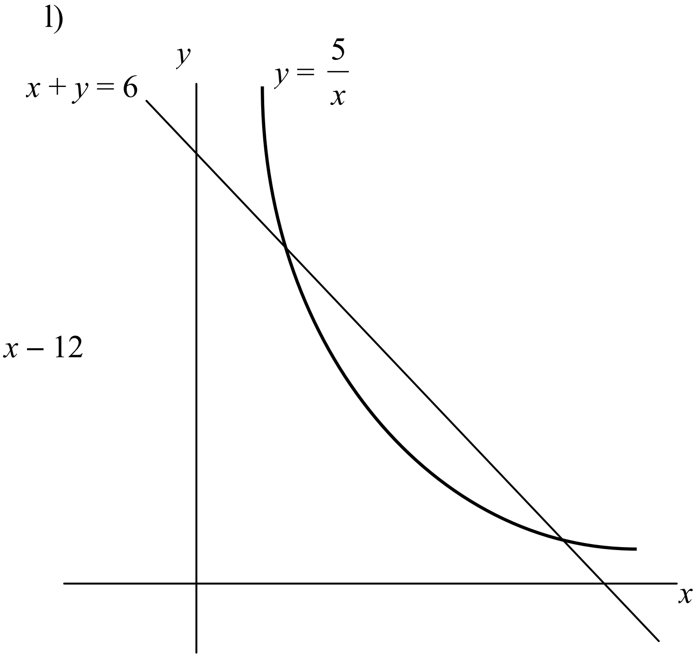
j)


y
y = 3x − 14
x
y = 8x − x2

k)


y
x
y = x − 12
y = 8 + 2x − x2

l)


y
x + y = 6
y = 5/x
x

8) Find the values, when they exist, of the following integrals.

a) $\int_1^\infty \frac{1}{x}\,dx$, b) $\int_1^\infty \frac{1}{x^{\frac{1}{3}}}\,dx$, c) $\int_1^\infty \frac{1}{x^3}\,dx$, d) $\int_0^2 \frac{1}{x^3}\,dx$,

e) $\int_0^2 \frac{1}{x^{\frac{1}{3}}}\,dx$, f) $\int_0^1 \frac{1}{x}\,dx$, g) $\int_1^\infty e^{(3-2x)}\,dx$.

9) Integrate the following with respect to x.

a) $\int (2 + e^{-x})\,dx$. b) $\int \frac{1}{2x+1}\,dx$. c) $\int (2x+1)^7\,dx$. d) $\int \frac{1}{1-x}\,dx$.

e) $\int (2x+3)^4\,dx$. f) $\int \left(1 + \frac{3}{4x}\right)dx$. g) $\int \left(x + 1 + \frac{1}{x}\right)dx$. h) $\int 6\sqrt{x}\,dx$.

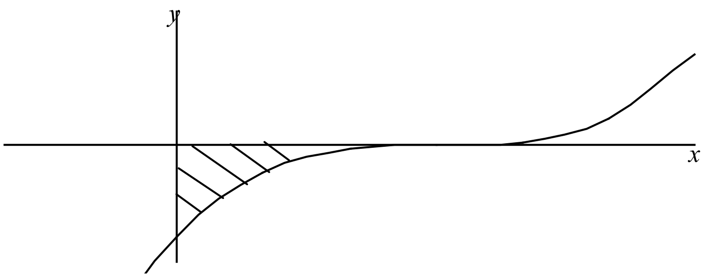
i) $\int \frac{1}{e^{2x}}\,dx$. j) $\int \frac{x^2 - 9}{x}\,dx$. k) $\int x(x^2 - 2)\,dx$. l) $\int \left(\frac{1}{x^3} + x^3\right)dx$.

10) Evaluate the following definite integrals.

a) $\int_1^4 6\sqrt{x}\,dx$. b) $\int_0^2 (x - \tfrac{1}{2}x^2)\,dx$. c) $\int_0^8 \frac{1}{\sqrt[3]{x}}\,dx$. d) $\int_0^2 \frac{1}{e^x}\,dx$. e) $\int_0^4 \frac{1}{1+2x}\,dx$.

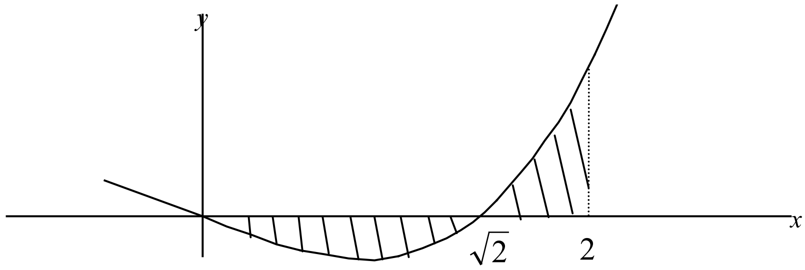
11) Find the exact value of $\int_0^\infty e^{1-3x}\,dx$.

12) The diagram shows the curve $y = (2x - 3)^3$.



The region shaded in the diagram is bounded by part of the curve and by the two axes. Find, by integration, the area of this region.
{**Hint:** find the x-coordinate of the point where the curve cuts the x–axis.}

13) The diagram shows the graph of $y = x(x^2 - 2)$ for $x \geq 0$.



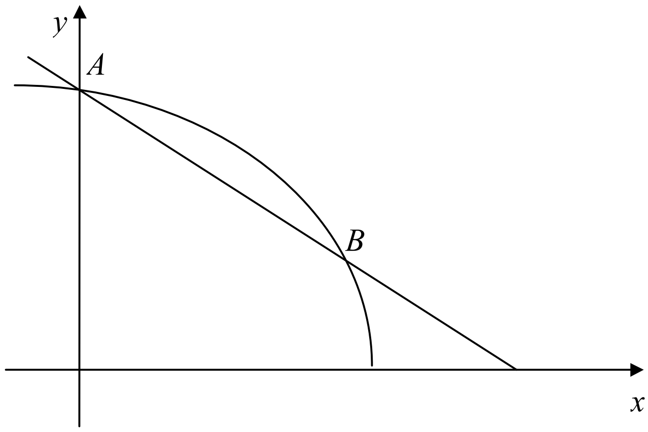
a) Find, by integration, the area of the shaded region.

{Hint: split the area into 2 parts, one positive and one negative.**}**

b) Find $\int_0^2 x(x^2 - 2)\,dx$.

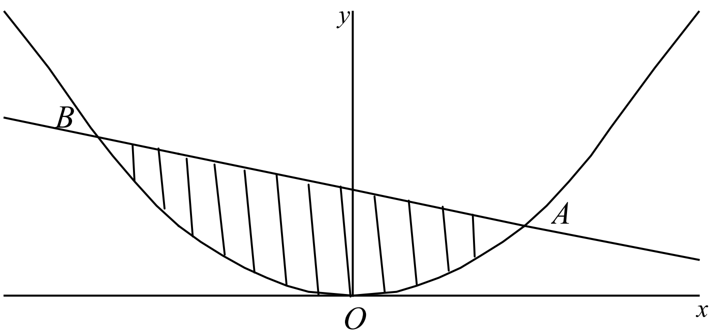
14) The diagrams shows a sketch of the curve $y = \sqrt{(4 - x)}$ and the line $y = 2 - \frac{1}{3}x$.

The co–ordinates of the points A and B where the curve and the line intersect are (0, 2) and (3, 1) respectively.



Calculate the area of the region between the line and the curve, giving your answer as an exact fraction.

15) The diagram shows a sketch of the graph of $y = x^2$ and the normal to the curve at the point $A(1, 1)$.

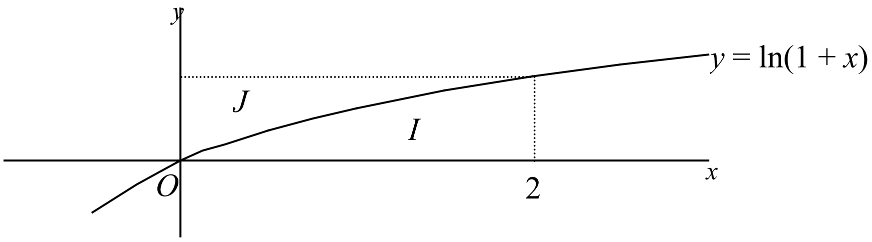


i) Use differentiation to find the equation of the normal **(not the *tangent*!)** at A. Verify that the point B where the normal cuts the curve again has co–ordinates $\left(-\frac{3}{2}, \frac{9}{4}\right)$.

{Hint: simply check that this point lies on the curve and the straight line.**}**

ii) The region which is bounded by the curve and the normal is shaded in the diagram. Calculate its area, giving your answer as an exact fraction.

*16) The diagram shows the curve $y = \ln(1 + x)$. The region bounded by the curve, the x–axis and the line $x = 2$ is denoted by I.



Rearrange $y = \ln(1 + x)$ to express x in terms of y.

Hence find the exact area of the region J shown in the diagram. **{Hint:** $\int_0^{\ln 3} x\,dy$. **}**

Deduce the area of the region I, giving the answer correct to 4 decimal places.

ANSWERS.

Section A. 1) $\frac{x^8}{8}+K$, 2) $\frac{3}{4}x^{\frac{4}{3}}+K$, 3) $\frac{4}{5}x^{\frac{5}{4}}+K$, 4) $\frac{-2}{\sqrt{x}}+K$, 5) $\frac{-1}{4x^4}+K$, 6) x^3-2x^2+x+K,
7) $\frac{2}{3}x^{\frac{3}{2}}+2\sqrt{x}+K$, 8) $\frac{2}{3}x^3-\frac{1}{x}+K$, 9) $x^5-x^4+x^2+K$, 10) $6x^{\frac{3}{2}}+K$, 11) $\frac{-2}{3}x^{-\frac{3}{2}}+K$, 12) $\frac{x^2}{4}-4\sqrt{x}+K$, 13) $\frac{2x^5}{5}+\frac{2}{3}x^{\frac{3}{2}}+K$, 14) $\frac{9}{2}x^{\frac{4}{3}}+K$, 15) $6x^{\frac{5}{3}}+K$, 16) $\frac{1}{6}(x+1)^6+K$, 17) $\frac{1}{8}(2x+1)^4+K$,
18) $\frac{1}{9}(3x-1)^3+K$, 19) $-\frac{1}{15}(5-3x)^5+K$, 20) $\frac{1}{6}(1+3x)^4+K$, 21) $-\frac{3}{25}(2-5x)^5+K$,
22) $-\frac{5}{9}(2-3x)^6+K$, 23) $\frac{2}{5}(1+5x)^4+K$, 24) $\frac{2}{3}(2+x)^{\frac{3}{2}}+K$, 25) $-\frac{1}{3}(5-2x)^{\frac{3}{2}}+K$,
26) $\frac{1}{4}(1+3x)^{\frac{4}{3}}+K$, 27) $-\frac{3}{8}(3-2x)^{\frac{4}{3}}+K$, 28) $\frac{2}{15}(3x+2)^{\frac{5}{2}}+K$, 29) $\sqrt{2x-4}+K$,
30) $\frac{3}{10}(5x-1)^{\frac{2}{3}}+K$, 31) $-\frac{1}{8(4x-1)^2}+K$, 32) $-\frac{3}{2(2x+1)}+K$, 33) $\frac{8}{3}\sqrt{3x-1}+K$, 34) $\frac{1}{(5-2x)^4}+K$,
35) $\frac{10}{3}\sqrt{x+1}+K$, 36) $\frac{1}{2}e^{2x}+K$, 37) $\frac{1}{3}e^{3x+1}+K$, 38) $2e^{3x-2}+K$, 39) $-\frac{1}{4}e^{2-4x}+K$, 40) $-\frac{1}{2}e^{-2x}+K$,
41) $-e^{-3x}+K$, 42) $-2e^{-5x}+K$, 43) $2e^{\frac{x}{2}}+K$, 44) $\ln(3+x)+K$, 45) $\ln(1+2x)+K$, 46) $4\ln(3+2x)+K$,
47) $\frac{1}{5}\ln(2+5x)+K$, 48) $\frac{3}{4}\ln(1+4x)+K$, 49) $-\ln(1-2x)+K$, 50) $-3\ln(2-3x)+K$,
51) $-\frac{3}{5}\ln(2-5x)+K$.

Section B.

1) a) $10\frac{1}{3}$, b) $\frac{1}{4}(e^4-1)$, c) $\ln\frac{5}{4}$, d) $12\frac{2}{3}$, e) 6, f) 2, g) 32.

2) a) 12 units^2, b) ln 4 units^2, c) $10\frac{2}{3}$ units^2, d) 128 units^2, e) 36 units^2, f) $\frac{1}{2}e^8+\frac{29}{6}$ or 1495·312327 units^2, g) $4\frac{1}{2}$ units^2.

3) 24. The integral equates to the area under the graph of $y=(2x+1)$ between $x=2$ and $x=5$ which is of course a trapezium etc.

4) 48 units^2.

5) Area = 21 units^2, –21.

6) i) $-\frac{1}{4}$, ii) $\frac{1}{4}$, iii) 0.

7) a) $20\frac{5}{6}$ units^2, b) $10\frac{2}{3}$ units^2, c) $20\frac{5}{6}$ units^2, d) 36 units^2, e) 36 units^2, f) $1\frac{1}{3}$ units^2, g) $121\frac{1}{2}$ units^2, h) $20\frac{5}{6}$ units^2, i) 36 units^2, j) $256\frac{1}{2}$ units^2, k) $121\frac{1}{2}$ units^2, l) 3·9528 units^2.

8) c) $\frac{1}{2}$, e) $\frac{3}{2}\times\sqrt[3]{4}$, g) $\frac{1}{2}e$. a), b), d) and f) do not exist.

9) a) $2x-e^{-x}+K$, b) $\frac{1}{2}\ln(2x+1)+K$, c) $\frac{1}{16}(2x+1)^8+K$, d) $-\ln(1-x)+K$, e) $\frac{1}{10}(2x+3)^5+K$, f) $x+\frac{3}{4}\ln x+K$, g) $\frac{1}{2}x^2+x+\ln x+K$, h) $4x^{\frac{3}{2}}+K$, i) $-\frac{1}{2}e^{-2x}+K$, j) $\frac{1}{2}x^2-9\ln x+K$, k) $\frac{1}{4}x^4-x^2+K$, l) $-\frac{1}{2x^2}+\frac{1}{4}x^4+K$.

10) a) 28, b) $\frac{2}{3}$, c) 6, d) $1-\frac{1}{e^2}$, e) ln 3.

11) $\frac{1}{3}e$.

12) $\frac{81}{8}$ units^2.

13) a) 2 units^2, b) 0.

14) $\frac{1}{6}$ units^2.

15) i) $y=-\frac{1}{2}x+\frac{3}{2}$, ii) $2\frac{29}{48}$ units^2.

16) $x=e^y-1$, area $J=(2-\ln 3)$ or 0·9013877 units^2, area $I=(3\ln 3-2)$ or 1·295836866 units^2.