

## INTEGRATION; Further Mathematics

1) Integrate the following by inspection or otherwise.

a)  $\int \frac{x}{4 - x^2} dx.$       b)  $\int \frac{x}{\sqrt{4 - x^2}} dx.$       c)  $\int \frac{e^{2x}}{1 + e^{2x}} dx.$       d)  $\int x^2 \sin x^3 dx.$

e)  $\int x^2 e^{x^3} dx.$       f)  $\int \frac{x}{\sqrt{x^2 - 1}} dx.$       g)  $\int \sin x \cos^4 x dx.$       h)  $\int x^2 \sqrt{2 + x^3} dx.$

i)  $\int \frac{x^2}{4 - x^3} dx.$       j)  $\int \frac{x e^{x^2}}{1 + e^{x^2}} dx.$       k)  $\int \frac{12x^2}{4 + x^3} dx.$       l)  $\int \frac{4 \cos x}{3 + \sin x} dx.$

m)  $\int \frac{\sec^2 x}{1 - \tan x} dx.$       n)  $\int \sinh x \cosh^3 x dx.$

2) Integrate the following by making a suitable substitution or otherwise.

a)  $\int x\sqrt{2 + x} dx.$       b)  $\int \frac{2x}{(1 + x^2)^3} dx.$       c)  $\int \cot x dx.$       d)  $\int x(x + 2)^4 dx.$

e)  $\int \sin x \sqrt{\cos x} dx.$       f)  $\int \tanh x dx.$       g)  $\int \cosh x \sinh 2x dx.$       h)  $\int_0^1 x^2 (1 - x)^{\frac{1}{2}} dx.$

i)  $\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x dx.$       j)  $\int_0^{\frac{\pi}{2}} \frac{\cos x}{4 + \sin x} dx.$       k)  $\int_0^{\frac{\pi}{3}} \sin 2x \sin x dx.$       l)  $\int_0^2 2x\sqrt{4 - x} dx.$

3) Find the following integrals by using either substitution or your formula book.

a)  $\int \frac{1}{\sqrt{1 - x^2}} dx,$       b)  $\int \frac{1}{1 + x^2} dx,$

c)  $\int \frac{1}{\sqrt{1 + x^2}} dx,$       d)  $\int \frac{1}{\sqrt{x^2 - 1}} dx,$

e)  $\int \frac{1}{\sqrt{4 - x^2}} dx,$       f)  $\int \frac{1}{9 + x^2} dx,$

g)  $\int \frac{1}{\sqrt{16 + x^2}} dx,$       h)  $\int \frac{1}{1 + 4x^2} dx,$

i)  $\int \frac{1}{\sqrt{16x^2 - 1}} dx,$       j)  $\int \frac{1}{\sqrt{4 - 9x^2}} dx.$

4) Find the following integrals by using the given substitution or otherwise.

[You will need the double-angle formulae for some of these!]

$$\begin{array}{ll}
 \text{a) } \int \frac{2x^2 - 1}{\sqrt{1-x^2}} dx, & x = \sin \theta, \\
 \text{b) } \int \sqrt{1-x^2} dx, & x = \sin \theta, \\
 \text{c) } \int \sin^3 x \cos^2 x dx, & u = \cos x, \\
 \text{d) } \int \cos \theta \sqrt{1 + \sin \theta} d\theta, & x = 1 + \sin \theta, \\
 \text{e) } \int \frac{dx}{x^2 \sqrt{9+x^2}}, & x = 3 \tan \theta, \\
 \text{f) } \int \sin 2\theta \sqrt{\sin \theta} d\theta, & u = \sin \theta, \\
 \text{g) } \int x \sqrt{x-1} dx, & u = x - 1, \\
 \text{h) } \int \frac{x^2}{\sqrt{4-x^2}} dx, & x = 2 \sin \theta, \\
 \text{i) } \int \frac{\cos \theta}{\sqrt{\sin \theta}} d\theta, & u = \sin \theta, \\
 \text{j) } \int \frac{dx}{(1-x^2)^{\frac{3}{2}}}, & x = \sin \theta, \\
 \text{k) } \int \frac{dx}{1+e^x}, & u = 1 + e^x, \\
 \text{l) } \int \frac{dx}{\sqrt{x^2+4}}, & x = 2 \sinh t.
 \end{array}$$

5) Use substitution or otherwise to integrate the following.

$$\begin{array}{lll}
 \text{a) } \int 4x \sqrt{1-x^2} dx, & \text{b) } \int x(x+1)^{\frac{1}{3}} dx, & \text{c) } \int \frac{\cos \theta}{1+\sin^2 \theta} d\theta, \\
 \text{d) } \int \frac{1}{16+9x^2} dx, & \text{e) } \int (\sqrt{x} + \frac{1}{\sqrt{x}})^3 dx, & \text{f) } \int \frac{dx}{e^x + 4e^{-x}}, \\
 \text{g) } \int x(1-x)^9 dx. & &
 \end{array}$$

6) By using the substitution  $x = 3 \cosh u$ , show that  $\int_3^5 \frac{1}{\sqrt{x^2-9}} dx = \ln 3$ .

7) By using the substitution  $x = 1 + \sinh u$ , show that  $\int_1^8 \frac{1}{\sqrt{x^2-2x+2}} dx = \ln(7+5\sqrt{2})$ .

8) By using the substitution  $x = \cosh u - 1$ , show that  $\int_0^{\frac{2}{3}} \frac{1}{\sqrt{x^2+2x}} dx = \ln 3$ .

9) a) Prove that  $\cosh^{-1}(x) = \ln(x + \sqrt{x^2-1})$ .

Hence show that  $\cosh^{-1}(\frac{3}{2}) = \ln(\frac{3+\sqrt{5}}{2})$  and find the exact value of  $\cosh^{-1}(\frac{5}{4})$ .

b) By using the substitution  $x = 4 \cosh u - 3$ , show that  $\int_2^3 \frac{1}{\sqrt{x^2+6x-7}} dx = \ln(\frac{3+\sqrt{5}}{4})$ .

ANSWERS.

- 1) a)  $-\frac{1}{2} \ln(4 - x^2) + K.$       b)  $-\sqrt{4 - x^2} + K.$       c)  $\frac{1}{2} \ln(1 + e^{2x}) + K.$   
d)  $-\frac{1}{3} \cos(x^3) + K.$       e)  $\frac{1}{3} e^{x^3} + K.$       f)  $\sqrt{x^2 - 1} + K.$   
g)  $-\frac{1}{5} \cos^5 x + K.$       h)  $\frac{2}{9} (2 + x^3)^{\frac{3}{2}} + K.$       i)  $-\frac{1}{3} \ln(4 - x^3) + K.$   
j)  $\frac{1}{2} \ln(1 + e^{x^2}) + K.$       k)  $4 \ln(4 + x^3) + K.$       l)  $4 \ln(3 + \sin x) + K.$   
m)  $-\ln(1 - \tan x) + K.$       n)  $\frac{1}{4} \cosh^4 x + K.$
- 2) a)  $\frac{2}{5} (2 + x)^{\frac{5}{2}} - \frac{4}{3} (2 + x)^{\frac{3}{2}} + K.$       b)  $-\frac{1}{2(1 + x^2)^2} + K.$   
c)  $\ln(\sin x) + K.$       d)  $\frac{(x + 2)^6}{6} - \frac{2(x + 2)^5}{5} + K.$   
e)  $-\frac{2}{3} (\cos x)^{\frac{3}{2}} + K.$       f)  $\ln(\cosh x) + K.$   
g)  $\frac{2 \cosh^3 x}{3} + K.$       h)  $\frac{16}{105}.$   
i)  $e - 1.$       j)  $\ln\left(\frac{5}{4}\right).$   
k)  $\frac{\sqrt{3}}{4}.$       l)  $\frac{256}{15} - \frac{112}{15} \sqrt{2} \quad (6.507205401).$
- 3) a)  $\sin^{-1} x + K.$       b)  $\tan^{-1} x + K.$       c)  $\sinh^{-1} x + K.$   
d)  $\cosh^{-1} x + K.$       e)  $\sin^{-1}\left(\frac{x}{2}\right) + K.$       f)  $\frac{1}{3} \tan^{-1}\left(\frac{x}{3}\right) + K.$   
g)  $\sinh^{-1}\left(\frac{x}{4}\right) + K.$       h)  $\frac{1}{2} \tan^{-1}(2x) + K.$       i)  $\frac{1}{4} \cosh^{-1}(4x) + K.$   
j)  $\frac{1}{3} \sin^{-1}\left(\frac{3x}{2}\right) + K.$
- 4) a)  $-x\sqrt{1 - x^2} + K.$       b)  $\frac{1}{2} x\sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x + K.$   
c)  $\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + K.$       d)  $\frac{2}{3} (1 + \sin \theta)^{\frac{3}{2}} + K.$   
e)  $\frac{-\sqrt{9 + x^2}}{9x} + K.$       f)  $\frac{4}{5} (\sin \theta)^{\frac{5}{2}} + K.$   
g)  $\frac{2}{5} (x - 1)^{\frac{5}{2}} + \frac{2}{3} (x - 1)^{\frac{3}{2}} + K.$       h)  $2 \sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{2} \sqrt{4 - x^2} + K.$   
i)  $2\sqrt{\sin \theta} + K.$       j)  $\frac{x}{\sqrt{1 - x^2}} + K.$   
k)  $x - \ln(1 + e^x) + K.$       l)  $\sinh^{-1}\left(\frac{x}{2}\right) + K.$

5) a)  $-\frac{4}{3}(1-x^2)^{\frac{3}{2}} + K.$

c)  $\tan^{-1}(\sin \theta) + K.$

e)  $\frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} + 6x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + K.$

g)  $\frac{(1-x)^{11}}{11} - \frac{(1-x)^{10}}{10} + K.$

b)  $\frac{3}{7}(x+1)^{\frac{7}{3}} - \frac{3}{4}(x+1)^{\frac{4}{3}} + K.$

d)  $\frac{1}{12}\tan^{-1}\left(\frac{3x}{4}\right) + K.$

f)  $\frac{1}{2}\tan^{-1}\left(\frac{e^x}{2}\right) + K.$