Forming and solving DIFFERENTIAL EQUATIONS

1) When a casserole was removed from an oven at 6 p.m., its temperature was 150°C. Its' temperature had fallen to 70°C by 6.20 p.m. The casserole can be eaten when its temperature has reached 30°C.

i) Estimate when this will be by assuming a constant rate of cooling from 150°C.

ii) An improved model of the casserole's temperature, θ°C, at time t minutes after 6 p.m., can be deduced from the assumptions that:

> the rate of loss of temperature is proportional to the difference in temperature between the casserole and the kitchen;
>
> the kitchen remains at a constant 20°C.

Show that these assumptions lead to an equation of the form $t = a\ln(\theta - 20) + b$, where a and b are constants.

Find a and b, and hence obtain a new estimate for the time at which the casserole can be put into the freezer.

2) In a certain pond, the rate of increase of the number of fish is proportional to the number of fish, n, present at time t. Assuming that n can be regarded as a continuous variable, write down a differential equation relating n and t and hence show that

$$n = Ae^{kt},$$

where A and k are constants.

In a revised model, it is assumed also that fish are removed from the pond, by anglers and natural wastage, at the constant rate of p per unit time, so that

$$\frac{dn}{dt} = kn - p.$$

Given that $k = 2$, $p = 100$ and that initially there were 500 fish in the pond, solve this differential equation, expressing n in terms of t.

Give a reason why this revised model is not satisfactory for large values of t.

3) During a spell of freezing weather, the ice on a pond has thickness x mm at time t hours after the start of freezing. at 3.00 p.m., after one hour of freezing weather, the ice is 2 mm thick and it is desired to predict when it will be 4 mm thick.

i) In a simple model, the rate of increase of x is assumed to be constant. For this model, express x in terms of t and hence determine when the ice will be 4 mm thick.

ii) In a more refined model, the rate of increase of x is taken to be proportional to $\frac{1}{x}$. Set up a differential equation for x, involving a constant of proportionality k.

Solve the differential equation and hence show that the thickness of ice is proportional to the square-root of the time elapsed from the start of freezing.

Determine the time at which the second model predicts that the ice will be 4 mm thick.

iii) What assumptions about the weather underlies both models.

4) A biologist studying fluctuations in the size of a particular population decides to investigate a model for which

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\cos kt,$$

where P is the size of the population at time t days and k is a positive constant.

i) Given that $P = P_0$ when $t = 0$, express P in terms of k, t and P_0.

ii) Find the ratio of the maximum size of the population to its minimum size.

5) In a certain type of chemical reaction a substance A is continuously transformed into a substance B. Throughout the reaction the sum of the masses A and B remains constant and equal to m. The mass of B present at time t after the commencement of the reaction is denoted by x.

At any instant the rate of increase of the mass of B is k times the mass of A, where k is a positive constant. Write down the differential equation relating x and t.

Solve this equation, given that $x = 0$ when $t = 0$.

Given also that $x = \frac{1}{2}m$ when $t = \ln 2$, determine the value of k, and show that at time t,

$$x = m(1 - e^{-t}).$$

Hence find: a) the value of x (in terms of m) when $t = 3\ln 2$.
b) the value of t when $x = \frac{3}{4}m$.

***{The following questions all involve *related rates of change*.)**

6) The initial volume of a spherical mothball is 5 cm^3. As it evaporates, its volume decreases.

a) It is given that the volume V cm^3 of the mothball remaining after time t weeks satisfies the differential equation

$$\frac{dV}{dt} = -\frac{1}{10}V.$$

Solve this differential equation to obtain an expression for the volume remaining after t weeks.

What does this model predict for:

i) the volume remaining after 1 week,

ii) the lifetime of the mothball?

b) In an improved model, it is assumed that the radius of the mothball decreases at a constant rate. Given that, after 1 week, the volume of the mothball is 4.5 cm^3, show that the lifetime of the mothball predicted by this improved model is approximately 29 weeks.

For this model, show that the rate of loss of volume is proportional to the mothball's surface area.

[For a sphere of radius r, surface area = $4\pi r^2$, volume = $\frac{4}{3}\pi r^3$.]

7) A rectangular water tank has a horizontal square base of side 1 metre. Water is flowing out of the tank from an outlet in the base but is also being pumped into the tank at a constant rate of 500 $\text{cm}^3\ \text{s}^{-1}$.
Initially, the depth of water in the tank is 121 cm and the water is flowing out at 1100 $\text{cm}^3\ \text{s}^{-1}$.

i) Assuming that the rate at which the water flows out remains constant, calculate the time, to the nearest second, taken for the depth of water to decrease from 121 cm to 100 cm.

ii) In an improved model, it is assumed that water flows from the outlet at a rate which is proportional to the square root of the depth, h cm, of water in the tank.

a) Show how this model leads to the differential equation $\frac{dh}{dt} = 0.05 - 0.01\sqrt{h}$,

where t is the time in seconds.

b) Show that the solution of the differential equation in a) is given by

$$t = \int \frac{100}{5-\sqrt{h}}\, dh.$$

Use the substitution $x = \sqrt{h} - 5$ to find the time, in seconds, taken for the depth of water to decrease from 121 cm to 100 cm.

8) A cylindrical water tank has a circular base of radius 1 metre. Water is flowing out of the tank from an outlet in the base but is also being pumped into the tank at a constant rate of 500 cm^3 s^{-1}. Initially, the depth of water in the tank is 121 cm and the water is flowing out at 1100 cm^3 s^{-1}.

i) Assuming that the rate at which the water flows out remains constant, calculate the time, to the nearest second, taken for the depth of water to decrease from 121 cm to 100 cm.

ii) In an improved model, it is assumed that water flows from the outlet at a rate which is proportional to the square root of the depth, h cm, of water in the tank.

a) Show how this model leads to the differential equation $\pi\dfrac{\mathrm{d}h}{\mathrm{d}t} = 0.05 - 0.01\sqrt{h}$, where t is the time in seconds.

b) Show that the solution of the differential equation in a) is given by

$$t = \int \frac{100\pi}{5-\sqrt{h}}\,\mathrm{d}h\,.$$

Use the substitution $x = \sqrt{h} - 5$ to find the time, in seconds, taken for the depth of water to decrease from 121 cm to 100 cm.

ANSWERS.

1) i) 6.30.

ii) Model gives $\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = -k(\theta - 20)$. Solve by separating the variables.

$a = -20.931199$. $b = 101.88333$. Estimated time = 6.54 p.m.

2) $\frac{\mathrm{d}n}{\mathrm{d}t} = kn$ for some constant k.

$n = 450e^{2t} + 50$.

This model predicts that the number of fish increases indefinitely which is unrealistic.

3) i) 4.00 p.m.

ii) $\frac{\mathrm{d}x}{\mathrm{d}t} = k \times \frac{1}{x}$. Solve by separating the variables and use the initial conditions; $t = 0, x = 0$ and $t = 1, x = 2$ to get $x = 2\sqrt{t}$.

This gives an estimated time of 6.00 p.m.

iii) We have assumed that the weather is either ‘freezing’ or ‘not freezing’. That is we are assuming that there is only one level of *freezing*.

4) i) $P = P_0 e^{\sin kt}$.

ii) Max $P = P_0 \times e$. Min $P = P_0 \times e^{-1}$. Ratio $= e^2$.

5) $\frac{\mathrm{d}x}{\mathrm{d}t} = k(m - x)$. Solve by separating the variables to get $\frac{m}{m - x} = e^{kt}$ or equivalent.

$k = 1$.

a) $x = \frac{7}{8}m$.

b) $t = \ln 4$ or $2\ln 2$.

6) a) $V = 5e^{-\frac{t}{10}}$. i) $V = 4.5$ cm^3. ii) Model predicts that the lifetime of the mothball is infinite(since the volume is never zero!)

b) Assume $\frac{\mathrm{d}r}{\mathrm{d}t} = -k$. This leads to $r = A - kt$ for some constant A. Using the fact that $V = \frac{4}{3}\pi r^3$ gives us that $V = \frac{4}{3}\pi(A - kt)^3$. Use the given information to find A and k etc.

For the final part, use the *chain rule* to get $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t}$ etc.

7) i) 350 seconds to the nearest second.

ii) a) We have $\frac{\mathrm{d}V}{\mathrm{d}t} = -k\sqrt{h} + 500$. Put $t = 0$, $h = 121$ and $\frac{\mathrm{d}V}{\mathrm{d}t} = -600$ to get $k = 100$.

Now use the *chain rule*: $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}h} \times \frac{\mathrm{d}h}{\mathrm{d}t}$ and rearrange etc.

b) {Don’t forget to include an arbitrary constant when integrating!}
Time taken is approximately 382 seconds.

8) This is very similar to question 7). The only real difference is that the formula for the volume of a cuboid is switched for the volume of a cylinder: $\pi r^2 h$.

i) 1100 seconds to the nearest second.
ii) b) 5465 seconds to the nearest second.