DETERMINANTS and INVERSES of 3×3 MATRICES

Matrices A, B, C and D are defined as follows:

$$A=\begin{bmatrix}1 & -1 & 3\\ 2 & 1 & 4\\ 0 & 1 & 1\end{bmatrix},\quad B=\begin{bmatrix}1 & 3 & -2\\ -2 & -9 & 5\\ 1 & 10 & 4\end{bmatrix},\quad C=\begin{bmatrix}3 & 2 & 6\\ 5 & 3 & 11\\ 7 & 4 & 16\end{bmatrix},\quad D=\begin{bmatrix}-4 & 3 & 5\\ 8 & 9 & -6\\ 2 & 6 & -1\end{bmatrix}$$

1. Find the determinants and, where they exist, the inverses of A, B, C and D.

2. Find the values of λ for which the matrix $\begin{bmatrix}6 & 7 & -1\\ 3 & \lambda & 5\\ 9 & 11 & \lambda\end{bmatrix}$ is singular (non-invertible).

3. The following questions verify general properties of determinants which can, if so desired, be memorised and used in an A-level exam. <u>They are not essential however</u>!

   i) <u>For any square matrix A, $\det A^{\mathrm{T}} = \det A$.</u> Verify by calculating $\det A^{\mathrm{T}}$ for the matrix A above.

   ii) <u>If 2 rows (or columns) of a square matrix A are equal, then $\det A = 0$.</u> Verify by calculating the determinants of
   $$\begin{bmatrix}1 & 2 & 3\\ 1 & 2 & 3\\ 0 & 1 & 1\end{bmatrix} \text{ and } \begin{bmatrix}1 & 1 & 3\\ 1 & 1 & 3\\ -2 & -2 & 1\end{bmatrix}.$$

   iii) <u>If 2 rows (or 2 columns) of a square matrix are interchanged, then its' determinant changes sign.</u>

   Verify by calculating the determinant of $\begin{bmatrix}2 & 1 & 4\\ 1 & -1 & 3\\ 0 & 1 & 1\end{bmatrix}$, obtained by switching the 1st and 2nd rows of the matrix A above.

   iv) <u>If one row (or column) of a square matrix A is multiplied by a constant λ, then the resulting determinant = $\lambda\det(A)$.</u>

   Verify by calculating the determinant of $\begin{bmatrix}2 & -2 & 6\\ 2 & 1 & 4\\ 0 & 1 & 1\end{bmatrix}$, obtained by doubling the first row of matrix A above.

v) <u>A determinant is unaffected if one row (or column) is added/subtracted from another row (or column).</u>

Verify by calculating the determinant of $\begin{bmatrix} 3 & 0 & 7 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{bmatrix}$, obtained from the matrix A (above) by adding the 2nd row to the 1st.

4. Prove that if A, B and C are invertible, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

5. Prove that if A is ANY 3×3 matrix, then $\det(2A) = 8\times\det(A)$.
{**Hint:** question 3 !}

ANSWERS.

1. $\det A = 5,\ A^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 & -7 \\ -2 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}$, $\det B = -25,\ B^{-1} = \frac{1}{25}\begin{pmatrix} 86 & 32 & 3 \\ -13 & -6 & 1 \\ 11 & 7 & 3 \end{pmatrix}$,

   $\det C = 0$, C^{-1} does not exist, $\det D = 30,\ D^{-1} = \frac{1}{30}\begin{pmatrix} 27 & 33 & -63 \\ -4 & -6 & 16 \\ 30 & 30 & -60 \end{pmatrix}$.

2. $\lambda = -2$ or 4.

3. i) $\det A^{\mathrm{T}} = 5 = \det A$ ii) both determinants = 0
   iii) determinant = $-5 = -\det A$ iv) determinant = $10 = 2\det A$
   v) determinant = $5 = \det A$.