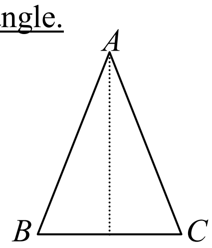
# BASIC GROUPS

1. For the following shapes; i) construct the Cayley table for the various symmetries under the operation of composition,

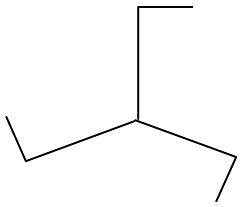
and ii) write down the inverse of each element.

a) <u>An isosceles triangle.</u>



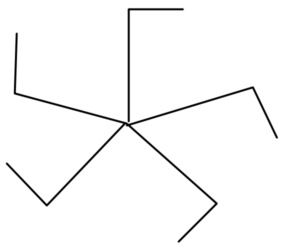

$e$ = the identity element,
$a$ = "reflection in the median through $A$"

b) <u>A 3-bladed windmill.</u>



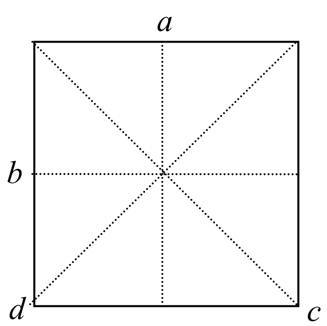
$e$ = the identity element
$r_1$ = "clockwise rotation of $120^{\circ}$"
$r_2$ = "clockwise rotation of $240^{\circ}$"

c) <u>A 5-sided windmill.</u>



$e$ = the identity element
$r_1$ = "clockwise rotation of $72^{\circ}$"
$r_2$ = "clockwise rotation of $144^{\circ}$"
$r_3$ = "clockwise rotation of $216^{\circ}$"
$r_4$ = "clockwise rotation of $288^{\circ}$"

d) <u>A square.</u>




$e$ = the identity element
$a$, $b$, $c$ and $d$ denote reflections in the lines shown.

$r$ = "clockwise rotation of $90^{\circ}$"
$s$ = "anti-clockwise rotation of $90^{\circ}$"
$t$ = "rotation of $180^{\circ}$"

2. Matrices $e$, $r$, $s$ and $t$ are defined as follows:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad r = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\quad s = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\quad \text{and}\quad t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Show that the set $\{e, r, s, t\}$ forms a group under the operation of matrix multiplication. You may assume associativity of matrix multiplication.
**{Hint: draw out the Cayley table and check all 4 of the group properties.}**

Is the group Commutative?

3. Prove that the set of transformations

$e$ = the identity
$a$ = "reflection in the $x$-axis"
$b$ = "reflection in the $y$-axis"
$c$ = "rotate through $180^{\circ}$ about the origin"

forms a group under the operation of composition. You may assume associativity.
**{Use matrices if you like!}**

4. Determine whether each of the following defines a group. Give reasons for your answers.

a) The set $\{0, 2, 4\}$ together with the operation of addition modulo 6, i.e. "add and take the remainder on dividing by 6".

b) The set $\{e, a, b\}$ together with the operation shown in the following Cayley table.

| | $e$ | $a$ | $b$ |
|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ |
| $a$ | $a$ | $e$ | $b$ |
| $b$ | $b$ | $a$ | $e$ |

**{Hint:** associativity!**}**

c) The set $\{0, 1, 2\}$ together with the operation of multiplication modulo 3, i.e. "multiply and take the remainder on dividing by 3".

d) The set $\{1, 2, 3, 4\}$ together with the operation of addition modulo 5.

5. A group of order 3 has elements $e$, $a$, and $b$, where $e$ denotes the identity element.

Use the fact that each element must occur exactly once in each column and row etc. to complete the Cayley table.

| | $e$ | $a$ | $b$ |
|---|---|---|---|
| $e$ | | | |
| $a$ | | | |
| $b$ | | | |

**{This shows that there is only one possible group of order 3 !}**

6. a) A group of order 4 has elements $e$, $a$, $b$, and $c$, where $e$ denotes the identity element.

Use the fact that each element must occur exactly once in each column and row etc. to complete the Cayley table.

| | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | | |
| $b$ | $b$ | | $e$ | |
| $c$ | $c$ | | | $e$ |

b) Repeat question a) for the following table.

| | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | | | $e$ |
| $b$ | $b$ | | $e$ | |
| $c$ | $c$ | $e$ | | |

**{a) and b) effectively show that there are only 2 possible groups of order 4 !}**

7. A group of order 6 has elements $I$, $P$, $Q$, $R$, $S$ and T, where $I$ denotes the identity element.

Use the fact that each element must occur exactly once in each column and row etc. to complete the Cayley table.

| | $I$ | $P$ | $Q$ | $R$ | $S$ | $T$ |
|---|---|---|---|---|---|---|
| $I$ | $I$ | $P$ | $Q$ | $R$ | $S$ | $T$ |
| $P$ | $P$ | $Q$ | $I$ | $T$ | | |
| $Q$ | $Q$ | | | $S$ | $T$ | $R$ |
| $R$ | $R$ | | | | $P$ | |
| $S$ | $S$ | | | $Q$ | | |
| $T$ | $T$ | $R$ | | | | |

8. Prove that the symmetry group $D_4$ of symmetries of the square is not commutative.
**{Use the Cayley table from question 1) d).}**

ANSWERS.

1. a)

| | $e$ | $a$ |
|---|---|---|
| $e$ | $e$ | $a$ |
| $a$ | $a$ | $e$ |

$e^{-1}=e$
$a^{-1}=a.$

b)

| | $e$ | $r_1$ | $r_2$ |
|---|---|---|---|
| $e$ | $e$ | $r_1$ | $r_2$ |
| $r_1$ | $r_1$ | $r_2$ | $e$ |
| $r_2$ | $r_2$ | $e$ | $r_1$ |

$e^{-1}=e$
$r_1^{-1}=r_2$
$r_2^{-1}=r_1$

c)

| | $e$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ |
|---|---|---|---|---|---|
| $e$ | $e$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ |
| $r_1$ | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $e$ |
| $r_2$ | $r_2$ | $r_3$ | $r_4$ | $e$ | $r_1$ |
| $r_3$ | $r_3$ | $r_4$ | $e$ | $r_1$ | $r_2$ |
| $r_4$ | $r_4$ | $e$ | $r_1$ | $r_2$ | $r_3$ |

$e^{-1}=e$
$r_1^{-1}=r_4$
$r_2^{-1}=r_3$
$r_3^{-1}=r_2$
$r_4^{-1}=r_1$

d)

| | $e$ | $r$ | $s$ | $t$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|---|---|---|
| $e$ | $e$ | $r$ | $s$ | $t$ | $a$ | $b$ | $c$ | $d$ |
| $r$ | $r$ | $t$ | $e$ | $s$ | $d$ | $c$ | $a$ | $b$ |
| $s$ | $s$ | $e$ | $t$ | $r$ | $c$ | $d$ | $b$ | $a$ |
| $t$ | $t$ | $s$ | $r$ | $e$ | $b$ | $a$ | $d$ | $c$ |
| $a$ | $a$ | $c$ | $d$ | $b$ | $e$ | $t$ | $r$ | $s$ |
| $b$ | $b$ | $d$ | $c$ | $a$ | $t$ | $e$ | $s$ | $r$ |
| $c$ | $c$ | $b$ | $a$ | $d$ | $s$ | $r$ | $e$ | $t$ |
| $d$ | $d$ | $a$ | $b$ | $c$ | $r$ | $s$ | $t$ | $e$ |

$e^{-1}=e$
$r_1^{-1}=r_4$
$r_2^{-1}=r_3$
$r_3^{-1}=r_2$
$r_4^{-1}=r_1$

2.

| | $e$ | $r$ | $s$ | $t$ |
|---|---|---|---|---|
| $e$ | $e$ | $r$ | $s$ | $t$ |
| $r$ | $r$ | $s$ | $t$ | $e$ |
| $s$ | $s$ | $t$ | $e$ | $r$ |
| $t$ | $t$ | $e$ | $r$ | $s$ |

The group is commutative.

3.

| | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

4. a) Yes. All 4 group properties are satisfied.

b) No since the element $b$ occurs twice in the final column. (Associativity fails.)

c) No since the element 0 has no inverse.

d) No, the closure property fails.

5.

| | $e$ | $a$ | $b$ |
|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ |
| $a$ | $a$ | $b$ | $e$ |
| $b$ | $b$ | $e$ | $a$ |

6. a)

| | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

b)

| | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $b$ | $c$ | $e$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $e$ | $a$ | $b$ |

7.

| | $I$ | $P$ | $Q$ | $R$ | $S$ | $T$ |
|---|---|---|---|---|---|---|
| $I$ | $I$ | $P$ | $Q$ | $R$ | $S$ | $T$ |
| $P$ | $P$ | $Q$ | $I$ | $T$ | $R$ | $S$ |
| $Q$ | $Q$ | $I$ | $P$ | $S$ | $T$ | $R$ |
| $R$ | $R$ | $S$ | $T$ | $I$ | $P$ | $Q$ |
| $S$ | $S$ | $T$ | $R$ | $Q$ | $I$ | $P$ |
| $T$ | $T$ | $R$ | $S$ | $P$ | $Q$ | $I$ |

8. E.g. $ar = c$ whilst $ra = d$ etc. Therefore $D_4$ is not commutative.