ALGEBRA IN GROUPS

{All groups are assumed to be finite.}

1. Prove that for any two elements x, y from a group G, $(xy)^{-1} = y^{-1}x^{-1}$.

2. i) Given that $ax = ay$ for elements a, x and y of a group G, show that $x = y$.

   ii) Given elements a and b of a group G, it is desired to find a 3rd element, x, such that $ax = b$.

G	.	x	.	.
.				
.				
a		b		
.				

   Show that there is only one unique member of G satisfying $ax = b$.
   {This proves that each element occurs <u>exactly</u> once in each row of the Cayley table !}

3. Given that $a^2b^2 = (ab)^2$ holds for each pair of elements a, b, in a group G, prove that G is commutative.

4. In a group G, it is known that each element is its' own inverse; i.e. $x^{-1} = x$ for every element x.

   Use the result of question 1 to prove that the group G must be commutative.

   {**Hint:** for elements a and b in the group G, $ab = (ab)^{-1}$ etc.}

5. For every element x in a group, it is known that $x^2 = e$.
   Prove that the group is commutative.

   {**Hint:** consider $(ab)^2 = (ab)(ab)$ etc. }

   NOTE THAT QUESTIONS 4 AND 5 ARE IDENTICAL!

6. Prove that there can only be one identity element in a group.
   {**Hint:** suppose there are 2 identities, e and f, and consider the product ef to show that e and f must be equal.}

7. A group G has 4 elements e, a, b and c, where e is the identity element.
   Prove that at least one of the elements a, b or c must be its' own inverse.
   {**Hint:** if $a^{-1} = b$ for example, then $b^{-1} = a$ etc.}

   Extend this result to show that any group G with an even order must have at least one element x such that $x^{-1} = x$.

Extension.

8. The sets $X = \{e, a, b\}$ and $Y = \{e, c, d\}$ are two subgroups of a group G.

   i) Prove that the product ac is not in the subgroup X.
   ii) Prove that the product ac is not in the subgroup Y.

   Deduce that the set $\{e, a, b, c, d\}$ is NOT a subgroup of G.

   {This is true of any 2 subgroups unless one is contained in the other.}

9. A multiplicative group has order 20, and the identity element is e. The 10 elements $\{e, g_1, g_2, \dots g_9\}$ form a subgroup G. The remaining 10 elements are $\{h_1, h_2, \dots h_{10}\}$, and this set is denoted by H.

   i) Give a simple reason why H cannot form a subgroup.

   ii) Show that the product of an element of G and an element of H (e.g. g_1h_1) cannot be in G, and hence must be in H.

   iii) Show that the inverse of any element in H must be in H.

10. Let G be a group with more than 2 elements in which every element x satisfies the equation $x^2 = e$.

    i) Show that G is commutative. **{See question 5.}**

    ii) Show that G contains a subgroup of order 4 and deduce that the number of elements of G is of the form $4k$, where k is an integer.

11. i) Use Lagrange's theorem to prove that any group whose order is a prime number must be cyclic.

    ii) G is a group of order n with no proper, non-trivial subgroups.
    Prove that G is cyclic and can be generated by ANY element other than the identity.

    Show also that if a number k divides n, the order of G, then G has a subgroup of order k.

    Hence deduce that n is a prime number.

12) The **centre**, C, of a group G is defined to be the set of elements of G which commute with every element of G, that is an element x is in C if $xg = gx$ for every element g in the group G.
    i) Show that the identity element, e, is in C.

    ii) Show that if elements x and y are in the centre C, then so is the product xy.

    iii) Show that if an element x is in the centre C, then so is its' inverse x^{-1}.

    This shows that the centre C is a subgroup of G.

HINTS.

1. Put $c = (xy)^{-1}$ which is equivalent to writing $c(xy) = e$.
   It is now a *simple* matter of showing that $c = y^{-1}x^{-1}$.

2. i) Easy!
   ii) If $ax = b$ then $x = a^{-1}b$ which proves the result.

3. $a^2b^2 = (ab)^2$ means that $a^2b^2 = abab$ etc. This simplifies to give $ab = ba$ as required.

4. $(ab) = (ab)^{-1} = b^{-1}a^{-1}$ (from 1) and thus $ab = ba$ as required.

5. $e = (ab)^2 = abab$ from which it is *easy* to show that $ab = ba$ as required.

6. Use the hint!

7. For the extension, consider pairing each element with its' inverse. If no element is its' own inverse (other than e) then there must be an odd number of elements! Why?

8. i) If ac is in X, then it is easy to show, using *closure*, that c must be in X.
   ii) Identical to i).
   $\{e, a, b, c, d\}$ is not closed since it does not contain ac. Thus it is not a subgroup.

9. i) H does not contain the identity.
   ii) See Q8 i).
   iii) Similar to Q8 i).

10. i) See Q5.

    ii) Since G has more than 2 elements, suppose that a, b are distinct elements of G not equal to the identity. It is not difficult to show that $\{e, a, b, ab\}$ are all distinct elements of G and forms a subgroup. For the last part think about Lagrange's theorem.

11. i) Consider the cyclic subgroup $\{x, x^2, x^3, \ldots\ldots x^k\}$ where x is not the identity element. Lagrange's theorem yields the fact that this subgroup must in fact be the entire group!

    ii) Very similar to i).

12. i) Obvious!

    ii) Easy!

    iii) $x^{-1}g = (g^{-1}x)^{-1}$ from question 1).
    $= (xg^{-1})^{-1}$ since x is in C.
    $= gx^{-1}$ from question 1).

    This shows that x^{-1} is in C.