SIMULTANEOUS EQUATIONS

Here lies one of the most technical topics to be encountered at GCSE, but one which tends to go down quite well with the majority of students. This is probably because there is a real sense of achievement in being able to solve these types of problems.

i) Algebraic solution.

A typical question involves us trying to find 2 unknown quantities; often labelled x and y.

For example, consider the equation $x + y = 10$ which asks:

"Which numbers, call them x and y, add up to give a total of 10 ?"

Well, we can have		$x = 10,$	$y = 0$
	or	$x = 9,$	$y = 1$
	or	$x = 8,$	$y = 2$
	or	$x = 7.8,$	$y = 2.2$
		etc.	

The point is that there are an unlimited number of such values for x and y, NOT a single pair of values !

Suppose then, that in addition to insisting that $x + y = 10$, we also insist that $x - y = 6$.
We are now asking:

"Which numbers, call them x and y, add up to give a total of 10 but at the same time have a difference of 6 ?"

This time there is only one possible pair of values for x and y, namely $x = 8$, $y = 2$.

We have just shown that the solution of the **simultaneous equations**

$$x + y = 10$$
$$x - y = 6$$

is $x = 8$ and $y = 2$.

And that is all there is to it. Well, almost all !

Example 1.

Solve the simultaneous equations $x + y = 12$

$x - y = 3.$

Comment.

We are being asked to find 2 values; x and y, which not only add up to 12, but at the same time differ by 3.

Well, listing all the obvious values adding up to 12 gives:

	$x = 12,\ y = 0$	a difference of 12
or	$x = 11,\ y = 1$	a difference of 10
or	$x = 10,\ y = 2$	a difference of 8
or	$x = 9,\ y = 3$	a difference of 6
or	$x = 8,\ y = 4$	a difference of 4
or	$x = 7,\ y = 5$	a difference of 2
etc.		

We cannot seem to find a pair which differ by 3. Still, this is a valid method of solution if we can locate the correct values.

Let us try some algebra instead.

Solution.

We have $x + y = 12$

and $x - y = 3.$

From the first equation we can rearrange to get $x = 12 - y$

From the second equation we can rearrange to get $x = 3 + y$

Now the only way that the value of x can equal both $12 - y$ and $3 + y$ is if these 2 quantities are equal.

Thus we have $12 - y = 3 + y$ which we solve to find the value of y.

$12 - y = 3 + y$

Add y to both sides to get $12 = 3 + 2y$

Subtract 3 from both sides $9 = 2y$

Now divide by 2 to get $y = 4.5$.

To finish, we know that $x = 3 + y$ and thus $x = 7.5$.

The solution to the simultaneous equations are thus given by $x = 7.5$ and $y = 4.5$.

NOTE.

There are literally a hundred different approaches to solving simultaneous equations, but all concentrate, to some extent, on *temporarily removing* one of the unknown quantities.

E.g. in our solution above we managed to remove the unknown x quantity, leaving us with the simple equation $12 - y = 3 + y$.

The point is that we cannot deal with 2 unknown quantities and must do whatever it takes to temporarily remove one of them.

Example 2.

Solve the simultaneous equations $3x + y = 10$
$2x - y = 10.$

Solution.

This time it looks as if we can quickly remove the unknown y value.

Rearrange the first equation to get $y = 10 - 3x$

Rearrange the second equation to get $2x = 10 + y$
and thus $2x - 10 = y.$

Hence we have $y = 10 - 3x$
and $y = 2x - 10$

This must mean that $10 - 3x = 2x - 10$ which we now solve to find x.

We have $10 - 3x = 2x - 10$

Add $3x$ to both sides to get $10 = 5x - 10$
Add 10 to both sides $20 = 5x$
and hence $x = 4$.

To finish we know that $y = 2x - 10$ and thus $y = 2 \times 4 - 10 = 8 - 10 = -2$.

The solution to the simultaneous equations are thus given by $x = 4$ and $y = -2$.

We thus see that it does not matter which unknown quantity; x or y, we choose to remove.

Example 3.

Solve the simultaneous equations $4x + 2y = 8$
$5x - 2y = 19.$

Solution.

This time it looks as if we can quickly remove the unknown y value. {Can you see why ?}

From the first equation we rearrange to get $2y = 8 - 4x$

From the second equation we rearrange to get $5x = 19 + 2y$
and thus $5x - 19 = 2y.$

Hence we have $2y = 8 - 4x$
and $2y = 5x - 19$

This must mean that $8 - 4x = 5x - 19$ which we now solve to find x.

We have $8 - 4x = 5x - 19$

Add $4x$ to both sides to get $8 = 9x - 19$
Add 19 to both sides $27 = 9x$

and hence $x = 3$.

To finish we know that $2y = 5x - 19$ and thus $2y = 5 \times 3 - 19 = 15 - 19 = -4$ from which we see that $y = -2$.

The solution to the simultaneous equations are thus given by $x = 3$ and $y = -2$.

By now, the reader should appreciate why I stated earlier that this is one of the most technical topics to be encountered at GCSE !

However, in an exam we might be looking at 4 marks, probably awarded as follows:

2 marks for correctly removing one of the unknowns
and 1 mark each for correctly identifying x and y.

The point is that with practice, the student should be looking to get at least 2 marks, probably even 3 marks on a regular basis. And on a good day ? Well, who knows !

It is certainly true that any student of mine failing to score above 2 marks on these would be expecting a good boot up the backside –and at no extra charge !

PRACTICE !

Example 4.

Solve the simultaneous equations $4x + 3y = 24$
$3x + 2y = 17.$

Comment.

This represents the most difficult question we are likely to face. The reason being that each equation has a different number of x's ($4x$ and $3x$) and a different numbers of y's and so it is not so easy to remove one of the unknown quantities.

Solution.

We choose to temporarily remove the x value. (It is just as *easy* to remove the y value.)

From the first equation we get $4x = 24 - 3y$
From the second equation we get $3x = 17 - 2y.$

Now since $4x$ and $3x$ are obviously different, we choose to *rescale* each equation in order to get $12x$'s:

Multiply both sides of $4x = 24 - 3y$ by 3 to get $12x = 72 - 9y$
Multiply both sides of $3x = 17 - 2y$ by 4 to get $12x = 68 - 8y$

{Now we are in business !}

This must mean that $72 - 9y = 68 - 8y$ which we solve to find y. {Take care with this as it involves lots of negatives !}

We have $72 - 9y = 68 - 8y$

Add $9y$ to both sides to get $72 = 68 + y$ {Since $-8y + 9y = y$}
and thus $y = 4$.

To find x, we might as well return to the very first equation which says:

$$4x + 3y = 24$$

and thus $4x + 3 \times 4 = 24$
or $4x + 12 = 24$
which means $4x = 12$
and hence $x = 3$.

The solution to the simultaneous equations are thus given by $x = 3$ and $y = 4$.

I agree, hard work this. The exercise below should keep you out of trouble for a while !

Exercise 1. **NO CALCULATORS.**

Solve the following pairs of simultaneous equations.

1) $2x + 3y = 11$
$5x + 3y = 23$

2) $5x + y = 12$
$4x + y = 10$

3) $4x + 3y = 18$
$2x + 3y = 12$

4) $5x + 6y = 31$
$4x + 6y = 26$

5) $3x + 5y = 19$
$2x + 5y = 16$

6) $4x + 6y = 16$
$7x + 6y = 19$

7) $3x + 4y = 20$
$5x + 4y = 28$

8) $8x + 3y = 19$
$5x + 3y = 13$

9) $5x + 2y = 22$
$3x + 2y = 14$

10) $5x + 4y = 24$
$7x + 4y = 32$

11) $5x + y = 17$
$4x - y = 10$

12) $4x + 3y = 22$
$5x - 3y = 14$

13) $5x + 4y = 14$
$3x - 4y = 2$

14) $4x + 5y = 17$
$4x - 5y = 7$

15) $6x + 4y = 16$
$3x - 4y = 2$

16) $5x - 3y = 18$
$2x + 3y = 24$

17) $4x - 5y = 7$
$2x + 5y = 11$

18) $3x - 2y = 14$
$4x + 2y = 28$

19) $2x - 5y = 11$
$3x + 5y = 29$

20) $4x - 7y = -3$
$3x + 7y = 10$

21) $2x + y = 9$
$3x + 2y = 14$

22) $5x + 3y = 16$
$x + y = 4$

23) $4x + 5y = 26$
$3x + y = 14$

24) $2x + 3y = 7$
$3x + 2y = 8$

25) $4x + 3y = 11$
$2x + 5y = 9$

26) $3x + 5y = 19$
$4x + 2y = 16$

27) $3x + 2y = 14$
$5x - 2y = 18$

28) $4x + 3y = 26$
$5x - 2y = 21$

29) $4x + 3y = 25$
$3x - 2y = 6$

30) $5x + 3y = 8$
$4x + 2y = 6.$

Answers.

1)	$x = 4$,	$y = 1$.	2)	$x = 2$,	$y = 2$.
3)	$x = 3$,	$y = 2$.	4)	$x = 5$,	$y = 1$.
5)	$x = 3$,	$y = 2$.	6)	$x = 1$,	$y = 2$.
7)	$x = 4$,	$y = 2$.	8)	$x = 2$,	$y = 1$.
9)	$x = 4$,	$y = 1$.	10)	$x = 4$,	$y = 1$.
11)	$x = 3$,	$y = 2$.	12)	$x = 4$,	$y = 2$.
13)	$x = 2$,	$y = 1$.	14)	$x = 3$,	$y = 1$.
15)	$x = 2$,	$y = 1$.	16)	$x = 6$,	$y = 4$.
17)	$x = 3$,	$y = 1$.	18)	$x = 6$,	$y = 2$.
19)	$x = 8$,	$y = 1$.	20)	$x = 1$,	$y = 1$.
21)	$x = 4$,	$y = 1$.	22)	$x = 2$,	$y = 2$.
23)	$x = 4$,	$y = 2$.	24)	$x = 2$,	$y = 1$.
25)	$x = 2$,	$y = 1$.	26)	$x = 3$,	$y = 2$.
27)	$x = 4$,	$y = 1$.	28)	$x = 5$,	$y = 2$.
29)	$x = 4$,	$y = 3$.	30)	$x = 1$,	$y = 1$.

ii) Graphical solution.

GCSE questions involving graphical solutions of simultaneous equations usually only require us to draw straight lines with relatively simple formulas.

Let us jump straight into an example.

Example 5.

Using axes: x from 0 to 6, y from −5 to 7, plot the graphs of

$y = x - 2$ and $2x + y = 7$ upon the same diagram.

Use your graphs to solve the simultaneous equations

$$y = x - 2$$
$$2x + y = 7.$$

Solution.

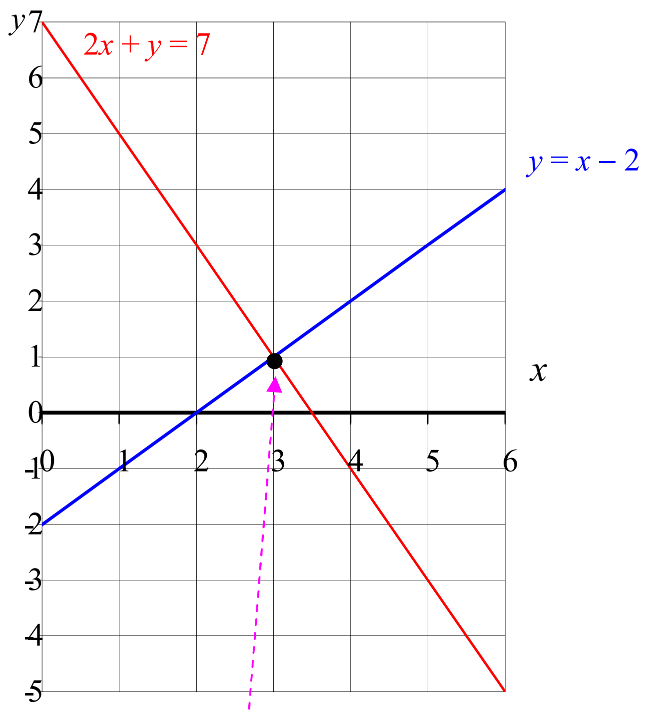
In plotting the graph of $y = x - 2$ we simply plot a few points (x, y) where the y-value is 2 less than the x-value. E.g. (2, 0), (3, 1), (4, 2) etc.

In plotting the graph of $2x + y = 7$ we first rearrange to get $y = 7 - 2x$.

Now draw up a table of values.

x	0	1	2	3	4
$y = 7 - 2x$	7	5	3	1	−1

We thus plot (0, 7), (1, 5) and (2, 3) etc. E.g. $x = 2$, $y = 7 - 2 \times 2 = 7 - 4 = 3$.



Now, the solution of the simultaneous equations

$$y = x - 2$$
$$2x + y = 7$$

is to be found at the point of intersection of the 2 graphs representing these equations.

The solution is thus given by $x = 3$ and $y = 1$.

Exercise 2. **NO CALCULATORS.**

1) **Axes:** x from 0 to 8, y from 0 to 10.
Plot the graphs of $x + y = 5$ and $y - x = 1$.

Use your graphs to solve the simultaneous equations
$$x + y = 5$$
$$y - x = 1.$$

2) **Axes:** x from -3 to 4, y from -2 to 13.
Plot the graphs of $x + y = 2$ and $y - 2x = 5$.

Use your graphs to solve the simultaneous equations
$$x + y = 2$$
$$y - 2x = 5.$$

3) **Axes:** x from -4 to 4, y from -7 to 3.
Plot the graphs of $y - x = -1$ and $y + x = -3$.

Use your graphs to solve the simultaneous equations
$$y - x = -1$$
$$y + x = -3.$$

4) **Axes:** x from -1 to 4, y from -3 to 12.
Plot the graphs of $y = x + 1$ and $3x + y = 9$.

Use your graphs to solve the simultaneous equations:
$$y = x + 1$$
$$3x + y = 9.$$

5) **Axes:** x from -3 to 6, y from 0 to 12.
Plot the following graphs of $y = 6 + x$ and $x + 2y = 6$.

Use your graphs to solve the simultaneous equations:
$$y = 6 + x$$
$$x + 2y = 6.$$

Answers.

1) $x = 2$, $y = 3$.
2) $x = -1$, $y = 3$.
3) $x = -1$, $y = -2$.
4) $x = 2$, $y = 3$.
5) $x = -2$, $y = 4$.

You might also consider checking your answers by solving each pair of simultaneous equations by using algebra. It's all good practice !